Approaching the Global Nash Equilibrium of Non-convex Multi-player Games

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Abstract—Many machine learning problems can be formulated as non-convex multi-player games. Due to non-convexity, it is challenging to obtain the existence condition of the global Nash equilibrium (NE) and design theoretically guaranteed algorithms. This paper studies a class of non-convex multi-player games, where players' payoff functions consist of canonical functions and quadratic operators. We leverage conjugate properties to transform the complementary problem into a variational inequality (VI) problem using a continuous pseudo-gradient mapping. We prove the existence condition of the global NE as the solution to the VI problem satisfies a duality relation. We then design an ordinary differential equation to approach the global NE with an exponential convergence rate. For practical implementation, we derive a discretized algorithm and apply it to two scenarios: multi-player games with generalized monotonicity and multi-player potential games. In the two settings, step sizes are required to be $\mathcal{O}(1/k)$ and $\mathcal{O}(1/\sqrt{k})$ to yield the convergence rates of $\mathcal{O}(1/k)$ and $\mathcal{O}(1/\sqrt{k})$, respectively. Extensive experiments on robust neural network training and sensor network localization validate our theory. Our code is available at https://github.com/GuanpuChen/Global-NE.

Index Terms—non-convex, multi-player game, Nash equilibrium, duality theory.

1 Introduction

Many advanced learning approaches in artificial intelligence are developed for multi-agent systems, distributed designs, or federated frameworks [1], [2], [3], [4]. As one of the most popular schemes, adversarial learning is gradually extended to involve multiple agents or players [5], [6], not restricted to classic models with one generator and one discriminator. Also, most complex systems involve the interaction and interference of multiple participants, such as smart grids [7], intelligent transportation [8], and cloud computing [9]. The core ideology is to utilize the autonomy of individual computing units in large-scale tasks. Traditional optimization frameworks or min-max adversarial protocols will no longer be universally applicable.

Game theory exploits the advantages in such multi-player scenarios and plays an essential role at the forefront of contemporary machine learning, such as adversarial training and reinforcement learning [10], [11], [12]. The Nash equilibrium (NE) [13] has become a popular concept in various fields like applied mathematics, computer sciences, and engineering, in addition to economics. When all players' strategies reach an NE, no one can benefit from changing their strategy unilaterally. This paper focuses on a typical class of non-convex multi-player games. Player $i \in \{1, \ldots, N\}$ minimizes its own payoff function $J_i(x_i, \boldsymbol{x}_{-i}) : \mathbb{R}^{Nn} \to \mathbb{R}$, which is influenced by both its own strategy $x_i \in \mathbb{R}^n$ and others' decisions $\boldsymbol{x}_{-i} \in \mathbb{R}^{(N-1)n}$. Specifically, player i's non-convex payoff function is

$$J_i(x_i, \boldsymbol{x}_{-i}) = \Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})).$$

Here, $\Lambda_i : \mathbb{R}^{Nn} \to \mathbb{R}^{q_i}$ is a vector-valued nonlinear operator, where $\Lambda_i = (\Lambda_{i,1}, \cdots, \Lambda_{i,q_i})^T$, and for $k \in \{1, \cdots, q_i\}$, $\Lambda_{i,k} : \mathbb{R}^{Nn} \to \mathbb{R}$ is a quadratic function in x_i. Besides, $\Psi_i : \mathbb{R}^{q_i} \to \mathbb{R}$ is a canonical function [14], whose gradient $\nabla \Psi_i$ is a one-to-one mapping from the primal space to the dual space.

This setting has been broadly investigated in various scenarios, including robust network training and sensor network localization. In sensor network localization [15], [16], for example, x_i is the location of an non-anchor node and $\Lambda_{i,k}$ represents the estimated distance between x_i and $\boldsymbol{x}_{-i}$. Also, Ψ_i is reified as the Euclidean norm to measure the distance errors. In robust neural network training [17], [18], x_i is the model parameter and $\Lambda_{i,k}$ serves as the output of training data. Also, Ψ_i represents the cross-entropy function. Moreover, this non-convex setting may also inspire solutions to resource allocation problems in unmanned vehicles [19] and secure transmission [20]. In this scenario, x_i stands for the transmit resources, Ψ_i denotes the transmission cost, and $\Lambda_{i,k}$ is a logarithmic-posynomial function.

Given the above formulation, it is essential to seek the global Nash Equilibrium (NE) from both game-theoretic and machine-learning perspectives. The equilibrium characterizes a global optimum solution, as no player will deviate from

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their strategy unilaterally given others' strategies. However, finding the global optimum or equilibrium in non-convex settings remains an open problem [21], [22]. This challenge arises from not only the lack of powerful tools compared to convex categories but also the diversity of non-convex structures, which may not be addressed by a common methodology. Despite the efficient tools available for convex conditions that have led to achievements in multi-player game models [23], [24], [25], they may fall short when dealing with non-convexity. Rather than reaching the global NE, these methods may become trapped in local NE or approximations when tracking along pseudo-gradients. On the other hand, inspiring breakthroughs have been made for solving two-player min-max games in various non-convex situations, such as Polyak-Łojasiewicz cases [17], [26] and concave cases [27], [28], Nonetheless, they may not be applicable to multi-player settings, because the global stationary conditions are interdependent and cannot be addressed individually by each player. Thus, novel processes are necessary to explore the existence of the global NE and design algorithms to approach it.

To address the non-convexity in payoff functions, we initially employ the canonical duality theory [14] and establish a one-to-one duality relation through a conjugate transformation [29]. By generalizing to continuous vector fields, we express the coupled stationary conditions of the transformed problem as a continuous mapping. Thus, finding the global NE of all players can be achieved by verifying a fixed point of this continuous mapping. We then cast the fixed point seeking problem to solve a variational inequality (VI) problem [30]. By following the above procedures, we can transform the global NE of a non-convex multi-player game into the solution to a VI problem. This simplifies the solving process, as all players' coupled stationary conditions are considered from a holistic perspective. Thereby, we can determine the existence condition of the global NE in such a non-convex multi-player game: the solution to the VI problem is required to satisfy the duality relation.

On this basis, we then propose a conjugate-based ordinary differential equation (ODE) to solve the VI problem. The ODE evolves in the dual spaces of both decision variables and canonical variables, rather than in the primal spaces. Next, we enforce a mapping from the dual space to the primal space using the gradient information from differentiable Legendre conjugate functions [31]. We demonstrate that the equilibrium of this conjugate-based ODE corresponds to the global NE of this non-convex multi-player game when the aforementioned existence condition is verified. Besides, we provide rigorous convergence analysis on the continuous dynamics and show that the ODE exhibits an exponential convergence rate.

For practical implementation, we further derive a discrete algorithm based on the proposed conjugate-based ODE. We analyze the step-size settings for achieving desired convergence rates in two typical non-convex game models. Specifically, with a step size of $\mathcal{O}(1/k)$, the convergence rate achieves $\mathcal{O}(1/k)$ in a class of multi-player games with generalized monotonicity [25], [32]; while with another step size of $\mathcal{O}(1/\sqrt{k})$, the convergence rate achieves $\mathcal{O}(1/\sqrt{k})$ in a class of multi-player potential games [15], [16].

We conduct extensive experiments in robust neural network training and sensor network localization. Our experimental results demonstrate that our algorithm converges to the global NE of non-convex games, avoiding being stuck in local NE or approximate NE. Furthermore, in order to show the superior performance, we compare our approach with several popular methods in these multi-player settings. Our code is available at https://github.com/GuanpuChen/Global-NE.

To the best of our knowledge, this is the first paper to address the existence condition and implementation algorithms for approaching the global NE in such a non-convex multi-player game. Our contributions are summarised below.

- **Existence condition.** We employ canonical duality theory to transform the non-convex multi-player game into a complementary dual problem. We convert solving the problem of all players' stationary point into solving a VI problem. We then establish the existence condition of the global NE: the solution to the VI problem is required to satisfy a duality relation.
- **Conjugate-based ODE.** We propose a conjugate-based ODE for solving the VI problem. The ODE evolves in the dual spaces of both decision variables and canonical variables. The equilibrium of the ODE is the global NE of this non-convex multi-player game, subject to the verification of the existence condition. The convergence analysis of the ODE is provided, together with its exponential convergence rate.
- **Discrete algorithm.** We derive a discrete algorithm based on the ODE for practical implementation. We apply it to two typical non-convex scenarios: 1) a convergence rate of $\mathcal{O}(1/k)$ in multi-player games with generalized monotonicity by a step size of $\mathcal{O}(1/k)$; 2) a convergence rate of $\mathcal{O}(1/\sqrt{k})$ in multi-player potential games by a step size of $\mathcal{O}(1/\sqrt{k})$.

Notations Let $\mathbb{R}^n$(or $\mathbb{R}^{m\times n}$) be the set of n-dimensional (or m-by-n) real column vectors (or matrices). Let $\mathbf{1}_n$(or $\mathbf{0}_n$) be the n-dimensional column vector with all elements of 1 (or 0). Take $\mathrm{col}\{x_1,\dots,x_n\} = (x_1^{\mathrm{T}},\dots,x_n^{\mathrm{T}})^{\mathrm{T}}$ and $\|\cdot\|$ as the Euclidean norm of vectors. Let ∇f denote function f's gradient and $\nabla_x f$ denote the partial derivative of function f in x. Additional, $\prod$ represents the product of all values in range series, $\succeq$ refers to positive semi-definiteness, and $\mathcal{O}$ describes the asymptotic upper bound of a function magnitude.

2 Related Work

Convex multi-player games: Many theoretical results in multi-player games have been built on fundamental convexity assumptions [23], [24], [25], [33]. Within the framework of convexity, extensive research has been conducted on seeking NE in various multi-player game models, including aggregative games [32], [34], potential games [16], [35], and sub-network games [36], [37]. For example, [25], [35] directly required convex payoffs on each player's decision variable, while [24], [32] needed strongly/strictly monotone pseudo-gradients to address the interaction of all players' decisions. Despite the efficient tools available within convex conditions that have led to fruitful achievements, they may fall short when encountering non-convexity in practical circumstances.

Non-convex two-player min-max problems: Significant breakthroughs have been made in solving non-convex two-player min-max problems, including Polyak-Łojasiewicz cases [17], [26], strongly-concave cases [27], [28], and general non-convex non-concave cases [38], [39]. The popularity of these approaches is attributed to the success of GANs and their variants [40], [41]. For instance, [38] introduced two time-scale update rules with stochastic gradient descent to find a local NE in GANs, while [39] developed an optimistic mirror descent algorithm to explore NE in GANs with theoretical guarantees. However, it is not straightforward and realistic to directly generalize the above two-player approaches to solve multi-player settings. This is due to the mutual coupling of global stationary conditions among multiple players, which cannot be handled individually, unlike two-player situations.

Non-convex multi-player games with local NE or approximations: Initial efforts have been made to solve non-convex multi-player games. [42] proposed a best-response scheme for Nash stationary points of a class of non-convex games in signal processing, and then [43] extended this method to multi-player bi-level games with non-convex constraints. Additionally, [44] introduced a gradient-based Nikaido-Isoda function to find Nash stationary points in a reformulated non-convex game, while [45] designed a gradient-proximal algorithm for approximate NE in a class of non-convex aggregative games. The algorithms within these works often result in local NE or Nash stationary points, which are dependent on the initial points. Further investigation is needed to guarantee the existence of the global NE and to design algorithms for seeking the global NE in non-convex multi-player game models.

Similar non-convex structures in optimization: Related results exist in solving such non-convex problems where the objectives or payoffs are composited of canonical functions and quadratic operators. However, these results are somewhat premature. [46] considered approximate optimization to relax such non-convex constraints and provided optimality conditions for the simplified problem, while [47] proposed similar sufficient conditions and discussed the existence of the global optimum using canonical duality theory. Building on this basis, [48] investigated the global optimal solution of such non-convex optimization problems in a distributed framework over multi-agent networks, while [49] focused on approximating the solutions of discrete variable topology problems with multiple constraints. We notice that, despite being regarded as an important class of non-convex problems, most existing work has focused on the optimization perspective. Nevertheless, considering the interference and interaction among multiple players, the global stationary conditions are coupled and can not be handled individually by each player. Therefore, the aforementioned optimization methods may not completely match our problems. The results in this paper will help elucidate the complex interactions among players and provide reliable insights for addressing large-scale game models in the future.

3 Preliminaries on Game Theory

We begin our study of the non-convex games with multiple players indexed by $\mathcal{I} = \{1, \cdots, N\}$. For $i \in \mathcal{I}$, the ith player has an action variable $x_i \in \mathbb{R}^n$ in an action set $\Omega_i \subseteq \mathbb{R}^n$, where Ω_i is compact and convex, and $\mathbf{\Omega} = \prod_{i=1}^N \Omega_i$. Let $\boldsymbol{x} = \mathrm{col}\{x_1, \cdots, x_N\} \in \mathbb{R}^{nN}$ be the profile of all players' actions, while $\boldsymbol{x}_{-i} = \mathrm{col}\{x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_N\} \in \mathbb{R}^{n(N-1)}$ be the profile of all players' actions except for the ith player's. Moreover, the ith player has a payoff function $J_i(x_i, \boldsymbol{x}_{-i}) : \mathbf{\Omega} \to \mathbb{R}$, which depends on both x_i and $\boldsymbol{x}_{-i}$, and is twice continuously differentiable with respect to x_i. Given $\boldsymbol{x}_{-i}$, the ith player aims to solve the following problem

$$\min_{x_i} J_i(x_i, \boldsymbol{x}_{-i}), \quad \text{s.t. } x_i \in \Omega_i. \tag{1}$$

In this paper, we focus on a typical class of non-convex multi-player games, where the payoff function of the ith player is endowed with the following structure

$$J_i(x_i, \boldsymbol{x}_{-i}) = \Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})). \tag{2}$$

Here $\Lambda_i : \mathbb{R}^{Nn} \to \Theta_i \subseteq \mathbb{R}^{q_i}$ is a vector-valued nonlinear operator with $\Lambda_i = (\Lambda_{i,1}, \cdots, \Lambda_{i,q_i})^T$. For $k \in \{1, \cdots, q_i\}$, each $\Lambda_{i,k} : \mathbb{R}^{Nn} \to \mathbb{R}$ is quadratic in x_i, whose second-order partial derivative in x_i is both x_i-free and $\boldsymbol{x}_{-i}$-free, e.g., $\Lambda_{i,k} = x_i^T A_{i,k} x_i + \sum_{i \neq j} x_i^T B_{i,k} x_j$. Moreover, $\Psi_i : \Theta_i \to \mathbb{R}$ is a convex differential canonical function [14], with its gradient $\nabla\Psi_i : \Theta_i \to \Theta_i^*$ being a one-to-one mapping. Such non-convex structures composited of canonical functions and quadratic operators arise in various applications, including robust network training [17], sensor localization [16], and GANs [50]. We provide specific examples in the following to illustrate the above non-convex model intuitively.

Example 1 (Euclidian distance function).

$$\sum\nolimits_{j \in \mathcal{N}_s^i} (\|x_i - x_j\|^2 - d_{i,j})^2, \tag{3}$$

where $\Psi_i = \sum_{j \in \mathcal{N}_s^i} \Lambda_{i,j}^T \Lambda_{i,j}$ and $\Lambda_{i,j} = \|x_i - x_j\|^2 - d_{ij}$. Function (3) usually serves as the payoff in sensor network localization [15], [16], [51], where $x_i \in \Omega_i$ is the location of non-anchor node i, $\mathcal{N}_s^i$ is the neighbors of node i, and d_{ij} describes the distance of nodes.

Example 2 (Log-sum-exp function).

$$\begin{aligned} &\log\Big[1 + \exp(-a_1 x_i^T A x_i - 2a_1 \sum\nolimits_{j \neq i, j=1}^N x_i^T A x_j \\ &-a_1 \sum\nolimits_{j \neq i, j=1}^N x_j^T A x_j - a_2 \beta_1^T x_i - \sum\nolimits_{j \neq i, j=1}^N a_2 \beta_1^T x_j)\Big], \end{aligned} \tag{4}$$

where $\Psi_i = \beta_1 \log[1 + \exp \Lambda_i]$ and $\Lambda_i = -a_1 x_i^T A x_i - 2a_1 \sum_{j \neq i, j=1}^N x_i^T A x_j - a_1 \sum_{j \neq i, j=1}^N x_j^T A x_j - a_2 \beta_1^T x_i - \sum_{j \neq i, j=1}^N a_2 \beta_1^T x_j$. Function (4) usually appears in the tasks like robust neural network training [18], [52], [53], where x_i is the neural network parameter and x_j is the perturbation. Then, a_1 and a_2 are positive parameters, while β_1 is the training data with $A = \beta_1 \beta_1^T$. More details of (4) can be found in Supplementary Materials.

Example 3 (Log-posynomial function).

$$\log(x_i^T \mathbf{C}_i x_i + x_i^T \mathbf{D}_i \boldsymbol{x}_{-i})^{-1}, \tag{5}$$

where $\Psi_i = \log(\Lambda_i)^{-1}$ and $\Lambda_i = x_i^T \mathbf{C}_i x_i + x_i^T \mathbf{D}_i \boldsymbol{x}_{-i}$. Function (5) usually occurs in resource allocation [19], [20], [54], where x_i stands for transmitting resources, and matrices $\mathbf{C}_i$ and $\mathbf{D}_i$ represent the correlation coefficients.

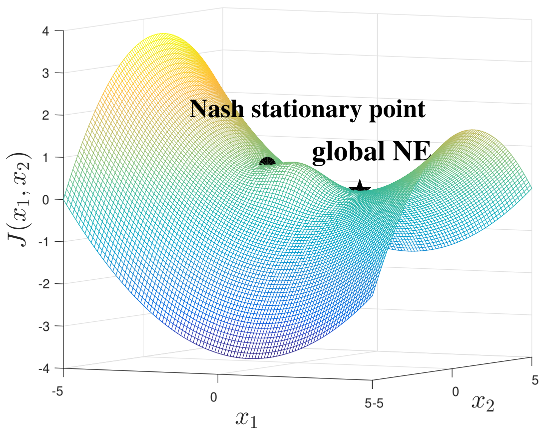


Fig. 1. A non-convex two-player demo with log-sum-exp payoffs in (4).

We introduce the following important concept for the non-convex multi-player game (1).

Definition 1 (global Nash equilibrium). A strategy profile $\boldsymbol{x}^\diamond \in \boldsymbol{\Omega}$ is said to be a global Nash equilibrium (NE) of (1), if for all $i \in \mathcal{I}$,

$$J_i(x_i^\diamond, \boldsymbol{x}_{-i}^\diamond) \leq J_i(x_i, \boldsymbol{x}_{-i}^\diamond), \quad \forall x_i \in \Omega_i. \tag{6}$$

The global NE characterizes a strategy profile in which each player adopts its globally optimal strategy. In other words, given others' actions, no player can benefit from unilaterally changing their decision. In fact, the conception of global NE discussed here is synonymous with the concept of NE [13], with the emphasis *global* in the non-convex formulation to distinguish it from *local* NE [17], [38], [42]. Also, we consider another well-known concept to help characterize the solutions to (1).

Definition 2 (Nash stationary point). A strategy profile $\boldsymbol{x}^\diamond$ is said to be a Nash stationary point of (1) if for all $i \in \mathcal{I}$,

$$\boldsymbol{0}_n \in \nabla_{x_i} J_i(x_i^\diamond, \boldsymbol{x}_{-i}^\diamond) + \mathcal{N}_{\Omega_i}(x_i^\diamond), \tag{7}$$

where $\mathcal{N}_{\Omega_i}(x_i^\diamond) = \{s \in \mathbb{R}^n : s^T(x - x_i^\diamond) \leq 0, \forall x \in \Omega_i\}$ is the normal cone at point $x_i^\diamond$ on set Ω_i.

It is not difficult to find that if $\boldsymbol{x}^\diamond$ is a global NE, then it must be a NE stationary point, but the converse is not necessarily true. For instance, in Fig. 1, the global NE differs from Nash stationary points, as shown on the surface plot of one player's non-convex payoff.

In the case of convex games, most existing research computes global NE by investigating Nash stationary points [24], [25], [32]. However, considering the bumpy geometric structure of the non-convex payoff function, as pointed out by [17], [55], one cannot expect to find a global NE of (1) only through the Nash stationary conditions in (7). Thereby, we aim at obtaining a global NE of such a non-convex multi-player model (1), and we begin the exploration in the sequel.

4 Existence Condition of Global NE

In this section, we primarily explore the existence of global NE through the following procedures:

i) We employ canonical duality theory to transform the original game (1) into a complementary dual problem and investigate the relationship between the stationary points of the dual problem and the Nash stationary points of game (1);
ii) We adopt a sufficient feasible domain for the introduced conjugate variable to investigate the global optimality of the stationary points;
iii) We cast the task of solving all players' stationary point profile of the dual problem into solving a variational inequality (VI) problem with a continuous pseudo-gradient mapping;
iv) We provide the existence condition of the global NE for the non-convex multi-player game: the solution to the VI problem is required to satisfy a duality relation.

Step 1: Complementary dual problem

We first take $\xi_i = \Lambda_i(x_i, \boldsymbol{x}_{-i}) \in \Theta_i$ in the payoff function of (2), which is referred to as a canonical measure, following the definition of canonical functions. Since $\Psi_i(\xi_i)$ is a convex canonical function, the one-to-one duality relation $\sigma_i = \nabla\Psi_i(\xi_i) : \Theta_i \to \Theta_i^*$ implies the existence of the conjugate function $\Psi_i^* : \Theta_i^* \to \mathbb{R}$, which can be uniquely described by the Legendre transformation [14], [29], [56].

$$\Psi_i^*(\sigma_i) = \xi_i^T \sigma_i - \Psi_i(\xi_i),$$

where $\sigma_i \in \Theta_i^*$ is a canonical dual variable. Take $\boldsymbol{\sigma} = \text{col}\{\sigma_1, \cdots, \sigma_N\}$ and $\boldsymbol{\Theta}^* = \prod_{i=1}^N \Theta_i^* \subseteq \mathbb{R}^q$ with $q = \sum_{i=1}^N q_i$. Then, the complementary function $\Gamma_i : \boldsymbol{\Omega} \times \Theta_i^* \to \mathbb{R}$ referring to the canonical duality theory can be defined as

$$\begin{aligned}\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}) =& \xi_i^T \sigma_i - \Psi_i^*(\sigma_i) \\ =& \sigma_i^T \Lambda_i(x_i, \boldsymbol{x}_{-i}) - \Psi_i^*(\sigma_i).\end{aligned} \tag{8}$$

Lemma 1. For a profile $\boldsymbol{x}^\diamond$, if there exists $\boldsymbol{\sigma}^\diamond \in \boldsymbol{\Theta}^*$ such that for $i \in \mathcal{I}$, $(x_i^\diamond, \sigma_i^\diamond)$ is a stationary point of complementary function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}^\diamond)$, then $\boldsymbol{x}^\diamond$ is a Nash stationary point of game (1).

Lemma 1 establishes the equivalency relationship of stationary points between (8) and (1). This indicates that we can close the duality gap between the non-convex original game and its canonical dual problem using the canonical transformation. More detailed proof of Lemma 1 can be found in the Supplementary Materials.

Step 2: Sufficient feasible domain

For $i \in \mathcal{I}$, the second-order partial derivative of $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ in x_i is

$$P_i(\sigma_i) = \nabla_{x_i}^2 \Gamma_i = \sum\nolimits_{k=1}^{q_i} [\sigma_i]_k \nabla_{x_i}^2 \Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}).$$

Recall that $\Lambda_i : \boldsymbol{\Omega} \to \Theta_i$ is a quadratic operator and $\nabla_{x_i}^2 \Lambda_i$ is both x_i-free and $\boldsymbol{x}_{-i}$-free (see the cases in (3)-(5)). Thus, $P_i(\sigma_i)$ is a linear combination of $[\sigma_i]_k$. On this basis, we introduce the following set of σ_i for $i \in \mathcal{I}$.

$$\mathscr{E}_i^+ = \Theta_i^* \cap \{\sigma_i : P_i(\sigma_i) \succeq \kappa_x \boldsymbol{I}_n\}, \tag{9}$$

where the constant $\kappa_x > 0$ and we further denote

$$\mathscr{E}^+ = \mathscr{E}_1^+ \times \cdots \times \mathscr{E}_N^+.$$

It follows from the compactness of Ω_i and Θ_i that Θ_i^* is compact. Thus, $\mathscr{E}_i^+$ is compact for $i \in \mathcal{I}$. When $\sigma_i \in \mathscr{E}_i^+$, the positive definiteness of $P_i(\sigma_i)$ implies that $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$

is convex with respect to x_i. Besides, the convexity of $\Psi_i(\xi_i)$ indicates that its Legendre conjugate $\Psi_i^*(\sigma_i)$ is also convex. Hence, the complementary function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ is concave in σ_i. This convex-concave property of Γ_i enables further investigation into the optimality of the stationary points of (8), namely, the optimality of the Nash stationary point of (1).

Remark 1. The computation of $\mathscr{E}_i^+$ is usually not so hard in most practical cases. For example, take the payoff function in (3) with $i = 1, 2$ and $n = q_i = 1$. Thus, the complementary function is $\Gamma_i(x_i, \sigma_i, x_{3-i}) = \sigma_i((x_i - x_{3-i})^2 - d_{i,3-i}) - \sigma_i^2/4$, where $x_i \in \Omega_i = [a, b]$ and $\sigma_i \in \Theta_i^* = [-2d_{i,3-i}, 2(b-a)^2 - 2d_{i,3-i}]$. Also, $\mathscr{E}_i^+ = \{\sigma_i : 2\sigma_i \geq \kappa_x\} \cap \Theta_i^* = [\kappa_x/2, 2(b-a)^2 - 2d_{i,3-i}]$, which can serve as the feasible constraint for the dual variable σ_i. We provide more examples in Supplementary Materials to show the derivation and computation of $\mathscr{E}_i^+$ in some large-scale problems.

Step 3: Variational inequality

Due to the interference of $\boldsymbol{x}_{-i}$, the transformed problem actually reflects a cluster of Γ_i with a mutual coupling of stationary conditions, rather than a deterministic one. Thus, unlike classic optimization works [14], [46], [47], the stationary points for player i cannot be calculated independently. We should consider the computation of all players' stationary points and discuss its optimality in an entire perspective.

Variational inequalities (VI) assist us in advancing [30]. Let $\boldsymbol{z} = \mathrm{col}\{\boldsymbol{x}, \boldsymbol{\sigma}\}$ and $\boldsymbol{\Xi} = \boldsymbol{\Omega} \times \mathscr{E}^+$. Denote the pseudo-gradient of (8) by the following continuous mapping

$$F(\boldsymbol{z}) = \mathrm{col}\Big\{ \mathrm{col}\{\textstyle\sum_{k=1}^{q_i} [\sigma_i]_k \nabla_{x_i} \Lambda_{i,k}(x_i, \boldsymbol{x}_{-i})\}_{i=1}^N, \\ \mathrm{col}\{-\Lambda_i(x_i, \boldsymbol{x}_{-i}) + \nabla \Psi_i^*(\sigma_i)\}_{i=1}^N \Big\}.$$

Note that the interaction of all players' variables is reflected in mapping F of the partial derivatives of all players' complementary functions (8). Then, solving the solution to (8) can be regarded as solving a VI problem $\mathrm{VI}(\boldsymbol{\Xi}, F)$, i.e., to find $\boldsymbol{z}^\diamond \in \boldsymbol{\Xi}$ such that

$$(\boldsymbol{z} - \boldsymbol{z}^\diamond)^T F(\boldsymbol{z}^\diamond) \geq 0, \quad \forall \boldsymbol{z} \in \boldsymbol{\Xi}. \tag{10}$$

Step 4: Existence condition

Based on the above steps, we obtain the following existence condition for identifying the global NE of (1).

Theorem 1. There exists $\boldsymbol{x}^\diamond$ as the global NE of the non-convex multi-player game (1) if $(\boldsymbol{x}^\diamond, \boldsymbol{\sigma}^\diamond)$ is a solution to $\mathrm{VI}(\boldsymbol{\Xi}, F)$ with $\sigma_i^\diamond = \nabla \Psi_i(\xi_i) \mid_{\xi_i = \Lambda_i(x_i^\diamond, \boldsymbol{x}_{-i}^\diamond)}$ for $i \in \mathcal{I}$.

The proof sketch can be summarized below. If there exists $\boldsymbol{\sigma}^\diamond \in \mathscr{E}^+$ such that $\boldsymbol{z}^\diamond = \mathrm{col}\{\boldsymbol{x}^\diamond, \boldsymbol{\sigma}^\diamond\}$ is a solution to $\mathrm{VI}(\boldsymbol{\Xi}, F)$, then it satisfies the first-order condition of the VI problem. Together with $\sigma_i^\diamond = \nabla \Psi_i(\xi_i) \mid_{\xi_i = \Lambda_i(x_i^\diamond, \boldsymbol{x}_{-i}^\diamond)}$, we claim that the canonical duality relation holds over $\Theta_i \times \mathscr{E}_i^+$ for $i \in \mathcal{I}$. It follows from Lemma 1 that the solution to $\mathrm{VI}(\boldsymbol{\Xi}, F)$ is a stationary point profile of (8) on $\Theta_i \times \Theta_i^*$. We can further verify that the complementary function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ is concave in dual variable σ_i and convex in x_i. In this light, we obtain the global optimality of $(\boldsymbol{x}^\diamond, \boldsymbol{\sigma}^\diamond)$ on $\boldsymbol{\Omega} \times \mathscr{E}^+$, that is, for $x_i \in \Omega_i$ and $\sigma_i \in \mathscr{E}_i^+$,

$$\Gamma_i(x_i^\diamond, \sigma_i, \boldsymbol{x}_{-i}^\diamond) \leq \Gamma_i(x_i^\diamond, \sigma_i^\diamond, \boldsymbol{x}_{-i}^\diamond) \leq \Gamma_i(x_i, \sigma_i^\diamond, \boldsymbol{x}_{-i}^\diamond).$$

This confirms that $\boldsymbol{x}^\diamond$ is the global NE of (1). The complete proof of Theorem 1 can be found in the Supplementary Materials.

The result in Theorem 1 reveals that once the solution of $\mathrm{VI}(\boldsymbol{\Xi}, F)$ is obtained, we can check whether the duality relation $\sigma_i^\diamond = \nabla \Psi_i(\xi_i) \mid_{\xi_i = \Lambda_i(x_i^\diamond, \boldsymbol{x}_{-i}^\diamond)}$ holds, so as to identify whether the solution of $\mathrm{VI}(\boldsymbol{\Xi}, F)$ is a global NE. Based on the above conclusion, we are motivated to solve $\mathrm{VI}(\boldsymbol{\Xi}, F)$ using its first-order conditions and employ the duality relation as a criterion for identifying the global NE.

Remark 2. The foundation for realizing the above idea is the nonempty set $\mathscr{E}_i^+$. It is possible to obtain an empty $\mathscr{E}_i^+$ in reality if $P_i(\sigma_i) \succeq \kappa_x \boldsymbol{I}_n$ has no intersection with Θ_i^*, rendering the above duality theory approach unavailable in such situations. Thus, $\mathscr{E}_i^+$ should be effectively checked once the problem is formulated. Such a process has also been similarly employed in some classic optimization works to solve non-convex problems [46], [47], [48], [49], [57]. In addition, this is why we cannot directly employ the standard Lagrange multiplier method and the associated KKT theory. We need to confirm a feasible domain of multiplier σ_i by utilizing canonical duality information (referring to Θ_i^*).

Remark 3. By generalizing the coupled stationary conditions to continuous vector fields, we compactly formulate these stationary conditions of the dual problem (8) as a continuous mapping in a VI problem $\mathrm{VI}(\boldsymbol{\Xi}, F)$. Thus, seeking all players' stationary points (or Nash stationary points) can be accomplished by verifying a fixed point. The seed of employing VI ideas in games dates back to [58], along with wide-ranging applications that can be found in a survey [59]. Moreover, non-convex formulation (1) may have more than one global NE due to the symmetry of $\Psi(\cdot)$. In such cases, the unique global NE can be regained by additionally introducing linear or quadratic terms as a perturbation to break the symmetry, which is commonly adopted in [15], [51].

5 Approaching Global NE via Conjugate-based ODE

In this section, we propose an ODE to seek the solutions to $\mathrm{VI}(\boldsymbol{\Xi}, F)$ (10) with the assistance of complementary information (the Legendre conjugate of Ψ_i and the canonical dual variable σ_i). An ODE provides continuously evolved dynamics, which help reveal how the primal variables and the canonical dual ones influence each other via conjugate gradient information. Moreover, analysis techniques in modern calculus and nonlinear systems for providing theoretical guarantees of ODEs may lead to results with mild assumptions.

5.1 ODE Design

Local set constraints of variables, like Ω_i and $\mathscr{E}_i^+$ of (10), are usually equipped with specific structures in various tasks. We employ conjugate properties of the generating functions within Bregman divergence to design ODE flows. Take $\phi_i(x_i)$ and $\varphi_i(\sigma_i)$ as two generating functions, where $\phi_i(x_i)$ is μ_x-strongly convex and L_x-smooth on Ω_i, and $\varphi_i(\sigma_i)$ is μ_σ-strongly convex and L_σ-smooth on $\mathscr{E}_i^+$. It follows from the

TABLE 1
Closed-form conjugate gradients with different generating functions.

	Feasible set	Generating function	Conjugate gradient
General convex set	Ω	$\frac{1}{2}\|x\|_2^2$	$\text{argmin}_{x\in\Omega}\frac{1}{2}\|x-y\|^2$
Non-negative orthant	$\mathbb{R}_+^n$	$\sum_{l=1}^n x^l\log(x^l)-x^l$	$\exp(y)$
Unit square $[a,b]^n$	$\{x^l\in\mathbb{R}: a\leq x^l\leq b\}$	$\sum_{l=1}^n(x^l-a)\log(x^l-a)$ $+(b-x^l)\log(b-x^l)$	$\text{col}\{\frac{a+b\exp(y^l)}{\exp(y^l)+1}\}_{l=1}^n$
Simplex Δ^n	$\{x\in\mathbb{R}_+^n: \sum_{l=1}^n x^l=1\}$	$\sum_{l=1}^n x^l\log(x^l)$	$\text{col}\{\frac{\exp(y^l)}{\sum_{j=1}^n\exp(y^j)}\}_{l=1}^n$
Euclidean sphere $\mathbf{B}_\rho^n(w)$	$\{x\in\mathbb{R}^n: \|x-w\|_2^2\leq p\}$	$-\sqrt{p^2-\|x-w\|_2^2}$	$py[\sqrt{1+\|y\|_2^2}]^{-1}-w$

Fenchel inequality [31] that the Legendre conjugate ϕ_i^* and φ_i^* are convex and differentiable. Specifically, for $y_i\in\mathbb{R}^n$,

$$\phi_i^*(y_i)\triangleq\min_{x_i\in\Omega_i}\{-x_i^Ty_i+\phi_i(x_i)\},$$

while for $\nu_i\in\mathbb{R}^{q_i}$,

$$\varphi_i^*(\nu_i)\triangleq\min_{\sigma_i\in\mathscr{E}_i^+}\{-\sigma_i^T\nu_i+\varphi_i(\sigma_i)\}.$$

Their conjugate gradients accordingly satisfy the relations

$$\nabla\phi_i^*(y_i)=\text{argmin}_{x_i\in\Omega_i}\left\{-x_i^Ty_i+\phi_i(x_i)\right\},\tag{11}$$

$$\nabla\varphi_i^*(\nu_i)=\text{argmin}_{\sigma_i\in\mathscr{E}_i^+}\{-\sigma_i^T\nu_i+\varphi_i(\sigma_i)\}.\tag{12}$$

On this basis, for player $i\in\mathcal{I}$, the conjugate-based ODE for seeking the global NE of the non-convex multi-player game (1) is

$$\begin{cases}\dot{y}_i=-\sigma_i^T\nabla_{x_i}\Lambda_i(x_i,\boldsymbol{x}_{-i})+\nabla\phi_i(x_i)-y_i\\ \dot{\nu}_i=\Lambda_i(x_i,\boldsymbol{x}_{-i})-\nabla\Psi_i^*(\sigma_i)+\nabla\varphi_i(\sigma_i)-\nu_i\\ x_i=\nabla\phi_i^*(y_i)\\ \sigma_i=\nabla\varphi_i^*(\nu_i).\end{cases}\tag{13}$$

The initial condition is $y_i(0)=y_{i0}\in\mathbb{R}^n$, $\nu_i(0)=\nu_{i0}\in\mathbb{R}^{q_i}$, $x_i(0)=\nabla\phi_i^*(y_{i0})$, and $\sigma_i(0)=\nabla\varphi_i^*(\nu_{i0})$. Here, t represents continuous time, and we drop t in the dynamics for a concise expression.

We give an explanation of two important operations for designing the conjugate-based ODE (13). On the one hand, we design the dynamics for $y_i(t)$ and $\nu_i(t)$ in dual spaces using the stationary conditions in (10). The dynamic updating of y_i and ν_i not only depends on player i's own decision but also requires knowledge of other players' decision $\boldsymbol{x}_{-i}$ contained in the function Λ_i and its partial derivative. According to Γ_i in (8), the terms $-\sigma_i^T\nabla_{x_i}\Lambda_i(x_i,\boldsymbol{x}_{-i})$ and $\Lambda_i(x_i,\boldsymbol{x}_{-i})-\nabla\Psi_i^*(\sigma_i)$ represent the directions of gradient descent and ascent, respectively. Besides, $\nabla\phi_i(x_i)$ and $\nabla\varphi_i(\sigma_i)$ are regarded as damping terms in ODE to avoid y_i and ν_i going to infinity [60], [61], [62]. On the other hand, the mapping from dual spaces back to primal spaces is implemented to update $x_i(t)$ and $\sigma_i(t)$ by virtue of conjugate gradients $\nabla\phi_i^*$ and $\nabla\varphi_i^*$, serving as the output feedback in system updating.

In fact, the conjugate-based idea not only helps solve the inherent non-convexity in such a class of multi-player games, but also brings the convenience of dealing with local set constraints with specific structures. It's important to note that the mappings via conjugate gradients $\nabla\phi_i^*$ and $\nabla\varphi_i^*$ are established based on valid generating functions rather than a conventional Euclidean norm. This yields explicit map relations between dual spaces and primal spaces to flexibly deal with diverse constraint conditions.

Remark 4. Without subscript i in Ω, we give some practical examples to show different closed-form conjugate gradients. When Ω is an n-dimensional unit simplex of soft-max output layers in GANs [41], i.e., $\Delta^n=\{x\in\mathbb{R}_+^n:\sum_{l=1}^n x^l=1\}$, a widely used generating function is the (negative) Gibbs–Shannon entropy $\phi(x)=\sum_{l=1}^n x^l\log(x^l)$, which yields the closed-form conjugate gradient $\nabla\phi^*(y)=\text{col}\{\frac{\exp(y^l)}{\sum_{j=1}^n\exp(y^j)}\}_{l=1}^n$. When Ω is a Euclidean sphere of parameter perturbation in adversarial training [18], i.e., $\mathbf{B}_\rho^n(w)=\{x\in\mathbb{R}^n:\|x-w\|_2^2\leq p\}$, the generating function can be chosen as $\phi(x)=-\sqrt{p^2-\|x-w\|_2^2}$, which explicitly yields $\nabla\phi^*(y)=py[\sqrt{1+\|y\|_2^2}]^{-1}-w$. Moreover, in sensor network localization [51], we can take $\phi(x)=(x-a)\log(x-a)+(b-x)\log(b-x)$ since $\Omega=[a,b]$ has a unit-square form. It brings the closed-form $\nabla\phi^*(y)=\text{col}\{\frac{a+b\exp(y^l)}{\exp(y^l)+1}\}_{l=1}^n$. Readers can check Table 1 for more cases.

5.2 Convergence Analysis

Next, we investigate the trajectories of the variables in(13) and analyze their convergence. Similarly to $\boldsymbol{x}$ and $\boldsymbol{\sigma}$, compactly denote $\boldsymbol{y}\in\mathbb{R}^{nN}$ and $\boldsymbol{\nu}\in\mathbb{R}^q$. Denote the profile of all Λ_i by $\Lambda(\boldsymbol{x})=\text{col}\{\Lambda_i(x_i,\boldsymbol{x}_{-i})\}_{i=1}^N$, and the augmented partial derivative profile by $G(\boldsymbol{x},\boldsymbol{\sigma})=\text{col}\left\{\sigma_i^T\nabla_{x_i}\Lambda_i(x_i,\boldsymbol{x}_{-i})\right\}_{i=1}^N$ .Then, ODE (13) can be compactly presented by

$$\begin{cases}\dot{\boldsymbol{y}}=-G(\boldsymbol{x},\boldsymbol{\sigma})+\nabla\boldsymbol{\phi}(\boldsymbol{x})-\boldsymbol{y}\\ \dot{\boldsymbol{\nu}}=\Lambda(\boldsymbol{x})-\nabla\Psi^*(\boldsymbol{\sigma})+\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma})-\boldsymbol{\nu}\\ \boldsymbol{x}=\nabla\boldsymbol{\phi}^*(\boldsymbol{y})\\ \boldsymbol{\sigma}=\nabla\boldsymbol{\varphi}^*(\boldsymbol{\nu}).\end{cases}\tag{14}$$

Lemma 2. Suppose that $(\boldsymbol{y}^\diamondsuit, \boldsymbol{\nu}^\diamondsuit, \boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ is an equilibrium point of ODE (14). If $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i)\mid_{\xi_i=\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)}$ for $i \in \mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of (1).

Lemma 2 establishes the relationship between the equilibrium of (14) (or (13)) and the global NE of (1). More proof details of Lemma 2 can be found in the Supplementary Materials due to space limitations.

The conjugate-based ODE (13) is designed to solve VI$(\boldsymbol{\Xi}, F)$ (10), i.e, the equilibrium point of (13) corresponds to the solution to (10). Recall the existence condition in Theorem 1. When the duality relation $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i)\mid_{\xi_i=\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)}$ is satisfied, we can derive that the equilibrium of ODE (13) realizes the global NE of non-convex multi-player game (1). The subsequent theorems present the main convergence results of ODE (13), implying that global NE can be found along the trajectory of ODE (13).

Theorem 2. If $\mathscr{E}_i^+$ is nonempty for $i \in \mathcal{I}$, then ODE (14) (or (13) is bounded and convergent. Moreover, if the convergent point $(\boldsymbol{y}^\diamondsuit, \boldsymbol{\nu}^\diamondsuit, \boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ satisfies $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i)\mid_{\xi_i=\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)}$ for $i \in \mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of (1).

In addition, given more information of function $\Psi_i(\cdot)$, we can derive the convergence rate of (13).

Theorem 3. For $i \in \mathcal{I}$, if $\mathscr{E}_i^+$ is nonempty and $\Psi_i(\cdot)$ is $\frac{1}{\kappa_\sigma}$-smooth, then (13) converges at an exponential rate with

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^\diamondsuit\| \leq \sqrt{\frac{\tau}{\mu}}\|\boldsymbol{z}(0)\| \exp(-\frac{\kappa}{2\tau}t),$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$, $\kappa = \min\{\kappa_\sigma, \kappa_x\}$, and $\tau = \max\{L_x/2\mu_x, L_\sigma/2\mu_\sigma\}$.

Here, we outline a proof for Theorems 2 and 3. We first prove that the trajectory $(\boldsymbol{y}(t), \boldsymbol{x}(t), \boldsymbol{\nu}(t), \boldsymbol{\sigma}(t))$ is bounded along ODE (13). To achieve this, we construct a Lyapunov candidate function using the Bregman divergence as follows

$$V_1 = \sum_{i=1}^N D_{\phi_i^*}(y_i, y_i^\diamondsuit) + D_{\varphi_i^*}(\nu_i, \nu_i^\diamondsuit).$$

We can carefully verify that

$$V_1 \geq \frac{\mu_x}{2}\left\|\boldsymbol{x} - \boldsymbol{x}^\diamondsuit\right\|^2 + \frac{\mu_\sigma}{2}\left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\right\|^2.$$

This means that V_1 is positive semi-definite, and is radially unbounded in $\boldsymbol{x}(t)$ and $\boldsymbol{\sigma}(t)$. The techniques used so far include the convexity of generating functions, the canonical dual relations, and the optimal conditions of VI problems, although the derivations are not straightforward. We next investigate the derivative of V_1 along the ODE. We can cautiously obtain that $dV_1/dt \leq 0$, which yields that the trajectories of $\boldsymbol{x}(t)$ and $\boldsymbol{\sigma}(t)$ are bounded along ODE (13). Likewise, we can conclude that $\boldsymbol{y}(t)$ and $\boldsymbol{\nu}(t)$ are also bounded. The above completes most parts in the proof of Theorem 2, except for some discussions on the invariant sets of the ODE. For Theorem 3, we can get $dV_1/dt \leq -\frac{\kappa}{\tau}V_1$ more than $dV_1/dt \leq 0$ thanks to the smoothness of function Ψ_i. This eventually results in the exponential rate. More detailed proofs of Theorems 2 and 3 can be found in the Supplementary Materials.

Remark 5. We summarize the road map for seeking global NE in this non-convex game for clarity. First, we should check whether $\mathscr{E}_i^+$ is nonempty upon formulating the problem. Next, we should seek the solution to VI$(\boldsymbol{\Xi}, F)$ via ODE flows, wherein the variable σ_i is restricted to the nonempty $\mathscr{E}_i^+$ and the implementation of the ODE is guaranteed. After obtaining the solution to VI$(\boldsymbol{\Xi}, F)$ by convergence, we should finally identify whether the convergent point satisfies the duality relation $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i)\mid_{\xi_i=\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)}$. If so, the convergent point is the global NE.

6 Discrete Algorithm

In this section, we consider deriving the discretization from the conjugate-based ODE for practical implementation. Notice that in each step of discrete algorithms, we can directly compute the minimum of a sub-problem, rather than track trajectories of conjugate functions with explicit expressions in the continuous ODE [61], [63]. Tapping into this advantage, the corresponding discrete algorithm is not obligated to resort to conjugate information like variables y_i and ν_i, simplifying the algorithm iteration.

6.1 Discrete Algorithm Design

We redefine an operator generated by Ψ_i on Θ_i by

$$\Pi_{\Theta_i}^{\Psi_i}(\sigma_i) = \text{argmin}_{\xi_i \in \Theta_i}\{-\sigma_i^T\xi_i + \Psi_i(\xi_i)\}.$$

This operator avoids computing the conjugate information of Ψ_i^*. Also, redefine operators $\Pi_{\Omega_i}^{\phi_i}(\cdot) = \nabla\phi_i^*(\cdot)$ in (11) and $\Pi_{\mathscr{E}_i^+}^{\varphi_i}(\cdot) = \nabla\varphi_i^*(\cdot)$ in (12). With a step size α_k at time k, we discretize the conjugate-based ODE (13) in the following.

Algorithm 1

Input: Step size $\{\alpha_k\}$, proper generating functions ϕ_i on Ω_i and φ_i on $\mathscr{E}_i^+$.

Initialize: $x_i^0 \in \Omega_i, \sigma_i^0 \in \mathscr{E}_i^+$, $i \in \{1, \ldots, N\}$.

1: **for** $k = 1, 2, \cdots$ **do**
2: **for** player $i \in \{1, \ldots, N\}$ **do**
3: compute the conjugate of Ψ_i: $\xi_i^k = \Pi_{\Theta_i}^{\Psi_i}(\sigma_i^k)$
4: update the decision variable : $x_i^{k+1} = \Pi_{\Omega_i}^{\phi_i}(-\alpha_k\sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k) + \nabla\phi_i(x_i^k))$
5: update the canonical dual variable: $\sigma_i^{k+1} = \Pi_{\mathscr{E}_i^+}^{\varphi_i}(\nabla\varphi_i(\sigma_i^k) + \alpha_k(\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k) - \xi_i^k))$
6: **end for**
7: **end for**

The update of variable x_i^{k+1} in Algorithm 1 can be equivalently expressed as

$$\text{argmin}_{x \in \Omega_i}\{\langle x, \sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k)\rangle + \frac{1}{\alpha_k}D_{\phi_i}(x, x_i^k)\},$$

where $D_{\phi_i}(x, x_i^k)$ is the Bregman divergence with generating function ϕ_i. A similar equivalent scheme can be found in σ_i^{k+1}. These equivalent iteration schemes reveal that parts of the idea in Algorithm 1, derived from the conjugate-based ODE (13), actually coincide with the *mirror descent* method [60]. Therefore, after computing the conjugate of Ψ_i and plugging it into the update of σ_i referring to properties of canonical functions and VIs, readers may also regard Algorithm 1 from the mirror descent perspective.

6.2 Step Size and Convergence Rate

Hereinafter, we provide the step-size settings and the corresponding convergence rates of Algorithm 1 in two typical non-convex multi-player games.

Multi-player generalized monotone games: Monotone games stand for a broad category in game models, where the pseudo-gradients of all players' payoffs satisfy the properties of monotonicity [24], [25], [32]. The monotone property yields the equivalence between the weak and strong solutions to VI problems [64], which makes most convex games solvable by the first conditions in VI. Analogously, we consider Algorithm 1 in a class of multi-player games with generalized monotonicity [59], referring to the canonical complementary function (8), and are rewarded by the following results.

Theorem 4. If $\mathscr{E}_i^+$ is nonempty and $\Pi_{\Theta_i}^{\Psi_i}(\cdot)$ is κ_σ-strongly monotone, then Algorithm 1 converges at a rate of $\mathcal{O}(1/k)$ with step size $\alpha_k = \frac{2}{\kappa(k+1)}$, i.e.,

$$\|\boldsymbol{x}^k - \boldsymbol{x}^\diamondsuit\|^2 + \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^\diamondsuit\|^2 \leq \frac{1}{k+1}\frac{M_1}{\mu^2\kappa^2},$$

where $\mu = \min\{\frac{\mu_x}{2}, \frac{\mu_\sigma}{2}\}$, $\kappa = \min\{\kappa_\sigma, \kappa_x\}$, and M_1 is a positive constant.

The proof sketch is presented as follows. Take a collection of the Bregman divergence as

$$\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \triangleq \sum_{i=1}^N D_{\phi_i}(x_i^\diamondsuit, x_i^{k+1}) + D_{\varphi_i}(\sigma_i^\diamondsuit, \sigma_i^{k+1}).$$

Here we employ the three-point identity, Fenchel's inequality, the strong monotonicity of $F(\boldsymbol{z})$, and the optimality of VI solution to process the above formula. See more details by Lemmas S3-S5 in the Supplementary Materials. Then, we get

$$\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \leq \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k\kappa\|\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit\|^2 + \frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2.$$

By substituting the above with $\eta_k = \kappa\alpha_k$, we take the sum of these inequalities over $k, \cdots, 1$ and obtain

$$\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \leq \eta_k^2(k+1)\frac{M_1}{4\kappa^2\mu}.$$

Owing to the step-size setting $\eta_k = \kappa\alpha_k = 2/(k+1)$, we finally reach the conclusion.

Multi-player potential games: Potential games also have a wide spectrum of applications such as power allocation [16], congestion control [35], and multi-target tracking [65]. In a potential game, there exists a unified potential function for all players, such that the change in each player's payoff is equivalent to the change in the potential function. Hence, the deviation in the payoff in (2) can be concretely mapped to a uniformed canonical potential function

$$J_i(x_i', \boldsymbol{x}_{-i}) - J_i(\boldsymbol{x}) = H(x_i', \boldsymbol{x}_{-i}) - H(\boldsymbol{x}). \tag{15}$$

Here, $H(\boldsymbol{x}) = \Psi(\Lambda(\boldsymbol{x}))$ is endowed with a canonical form. With a common canonical dual variable σ, the complementary function is

$$\Gamma_i(x_i, \sigma, \boldsymbol{x}_{-i}) = \Gamma(\boldsymbol{x}, \sigma) = \sigma^T\Lambda(\boldsymbol{x}) - \Psi^*(\sigma). \tag{16}$$

Also, the set $\mathscr{E}^+$ of σ is in a unified form similar to (9). Take the weighted averages $\hat{\boldsymbol{x}}^k$ and $\hat{\sigma}^k$ in the course of k iterations as $\hat{\boldsymbol{x}}^k = \frac{\sum_{j=1}^k \alpha_j\boldsymbol{x}^j}{\sum_{j=1}^k \alpha^j}$ and $\hat{\sigma}^k = \frac{\sum_{j=1}^k \alpha_j\sigma^j}{\sum_{j=1}^k \alpha^j}$, respectively. We give the convergence rate of Algorithm 1 as below.

Theorem 5. If $\mathscr{E}^+$ is nonempty and players' payoffs are subject to the potential function in (15), then Algorithm 1 converges at a rate of $\mathcal{O}(1/\sqrt{k})$ with step size $\alpha_k = \frac{2\mu d}{M_2\sqrt{k}}$, i.e.,

$$\Gamma(\hat{\boldsymbol{x}}^k, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \hat{\sigma}^k) \leq \frac{1}{\sqrt{k}}\sqrt{\frac{d}{\mu}}M_2,$$

where $\mu = \min\{\frac{\mu_x}{2}, \frac{\mu_\sigma}{2}\}$, and d and M_2 are two positive constants.

We also show an outline of the proof. Take another collection of the Bregman divergence as

$$\widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}) \triangleq D_\varphi(\sigma^\diamondsuit, \sigma) + \sum_{i=1}^N D_{\phi_i}(x_i^\diamondsuit, x_i).$$

By the three-point identity and $\sigma \in \mathscr{E}^+$ in (9), the VI with respect to $\boldsymbol{z}^k$ can be bounded by the duality gap of the complementary function, that is,

$$\left\langle F(\boldsymbol{z}^k), \boldsymbol{z}^\diamondsuit - \boldsymbol{z}^k \right\rangle \leq \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^k) - \Gamma(\boldsymbol{x}^k, \sigma^\diamondsuit).$$

Then, we can further derive over $1, \cdots, k$ that

$$\sum_{j=1}^k \alpha_j\left(\Gamma(\boldsymbol{x}^j, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^j)\right) \leq \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^1) + \frac{\sum_{j=1}^k \alpha_j^2 M_2^2}{4\mu}.$$

Owing to the Jensen's inequality and the step-size setting $\alpha_k = 2\sqrt{\mu d}/M_2\sqrt{k}$, we finally reach the conclusion. The whole proof can be found in the Supplementary Materials.

Remark 6. In the aforementioned two cases, our algorithm performs with the best-known convergence rates and aligns with the results under convex circumstances. Under strongly monotone conditions, our algorithm for non-convex settings achieves the same convergence rate of $\mathcal{O}(1/k)$ as convex cases [66], [67]. The proof is established on the scales from variational theory and the measurement of Bregman divergence, which reflects the convergence rate with respect to the equilibrium point. Regarding potential games, we utilize the unified potential function to derive the same convergence rate of $\mathcal{O}(1/\sqrt{k})$ as convex cases [68], [69]. The convergence result is described by the duality gap within the potential function correspondingly.

7 Experiments

In this section, we evaluate the effectiveness of our approach for seeking the global NE in the practical tasks of robust neural network training and sensor network localization. Our code is available at https://github.com/GuanpuChen/Global-NE.

7.1 Robust neural network training

In this part, we show the convergence performance of our approach under a robust neural network circumstance.

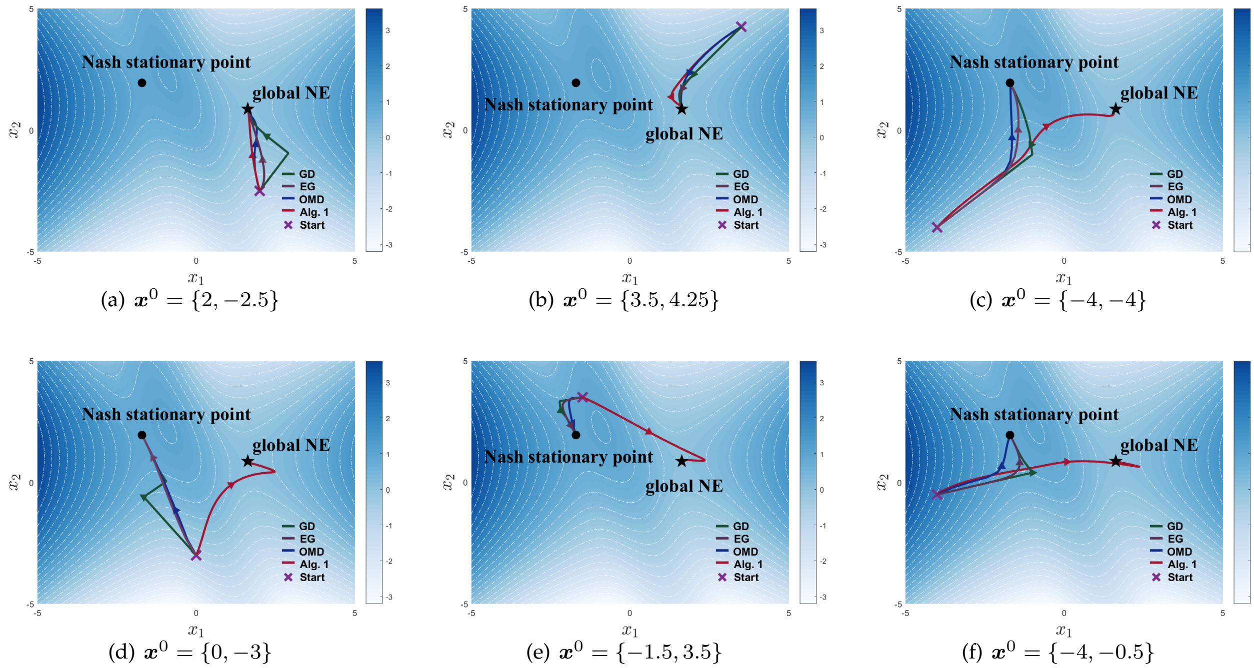


Fig. 2. Performance of different methods with different initial points.

Model and data pre-processing. Consider an adversarial training task [17], [18], [52], where the min-max payoff for two players is

$$\min_{x_1\in\Omega_1}\mathbb{E}_{(\beta_0,\beta_2)\sim\mathcal{D}}\max_{x_2\in\Omega_2} Loss(x_1,x_2,\beta_1,\beta_2)+\frac{\lambda_1}{2}\|x_1\|^2-\frac{\lambda_2}{2}\|x_2\|^2. \tag{18}$$

In the above expression (18), $Loss$ refers to the cross entropy loss function, where x_1 is the neural network parameter, x_2 is the parameter perturbation, and β_1, β_2 are perturbed training data. $\mathcal{D}$ is the local data distribution. Here, β_1 is an adversarial example of a clean one β_0 with a perturbation ϵ, while β_2 is the real label of β_0. The last two terms of (18) are regularizers.

Denote $(\beta_{1,l}, \beta_{2,l}, \beta_{0,l})$ as the l-th data pair. The explicit form is $Loss = -(\beta_2 \log(s) + (1 - \beta_2)\log(1 - s))$, where $s = \sigma(z) = 1/(1+\exp(-a_1 z^2 - a_2 z))$ is a quadratic sigmoid activation function [70] with two parameters $a_1, a_2 > 0$, and $z = (x_1 + x_2)^T\beta_1$ is the output. With $A_l = \beta_{1,l}\beta_{1,l}^T$,

$$\begin{aligned}Loss=&\frac{1}{q}\sum_{l=1}^{q}\log[1+\exp(-a_1x_1^TA_lx_1-2a_1x_1^TA_lx_2-a_1x_2^TA_lx_2\\&-a_2\beta_{1,l}^Tx_1-a_2\beta_{1,l}^Tx_2)]+(1-\beta_{2,l})\big(a_1x_1^TA_lx_1+2a_1x_1^TA_lx_2\\&+a_1x_2^TA_lx_2+a_2\beta_{1,l}^Tx_1+a_2\beta_{1,l}^Tx_2\big).\end{aligned} \tag{19}$$

The goal of the adversarial learning task is to improve the robustness of neural networks against adversarial examples.

We will illustrate the performance via both artificial data and real data. Artificial data is generated to examine a one-dimensional adversarial training problem where the constraints are subject to local bounds. Also, we consider two real datasets. One is MNIST [71], containing 784 features. The other is used for identifying cats from h5py library [72], containing 12288 features. We consider the binary classification problem and divide the labels of MNIST into zero and non-zero and the other into cat and non-cat. Then, we standardize the data using min-max normalization. Also, we will compare the performance of Algorithm 1 in this task with some state-of-the-art algorithms [17], [18], [52].

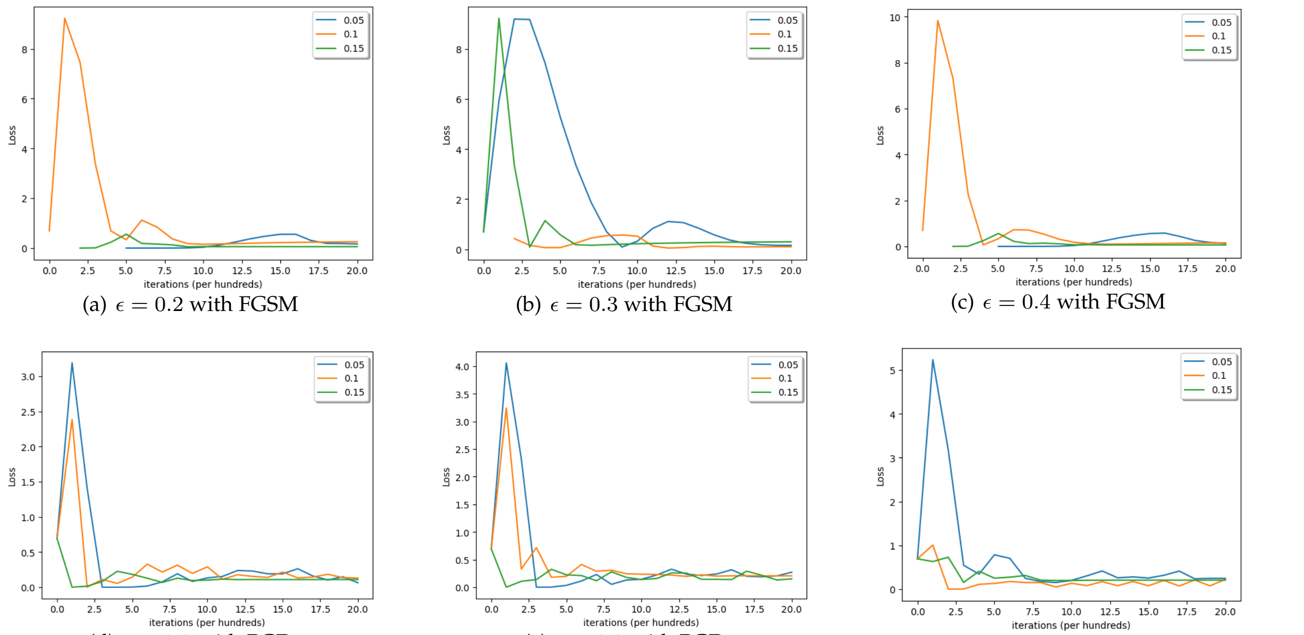
Method. We obtain the complementary function in (17) and employ Algorithm 1 to solve this problem. Based on (9), we can derive

$$\mathscr{E}^+=\Theta^*\cap\{\sigma_1:\frac{\lambda_1-\kappa_x}{\omega_{l1}}\boldsymbol{I}_n\geq\sigma_{1,l}\geq\frac{\kappa_x-\lambda_2}{\omega_{l2}}\boldsymbol{I}_n, l=1,\cdots,q\},$$

where ω_{l1} and ω_{l2} are the largest and the smallest eigenvalues of A_l, respectively. Set tolerance $t_{\text{tol}} = 10^{-3}$ and the terminal criterion $\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\| \leq t_{\text{tol}}$.

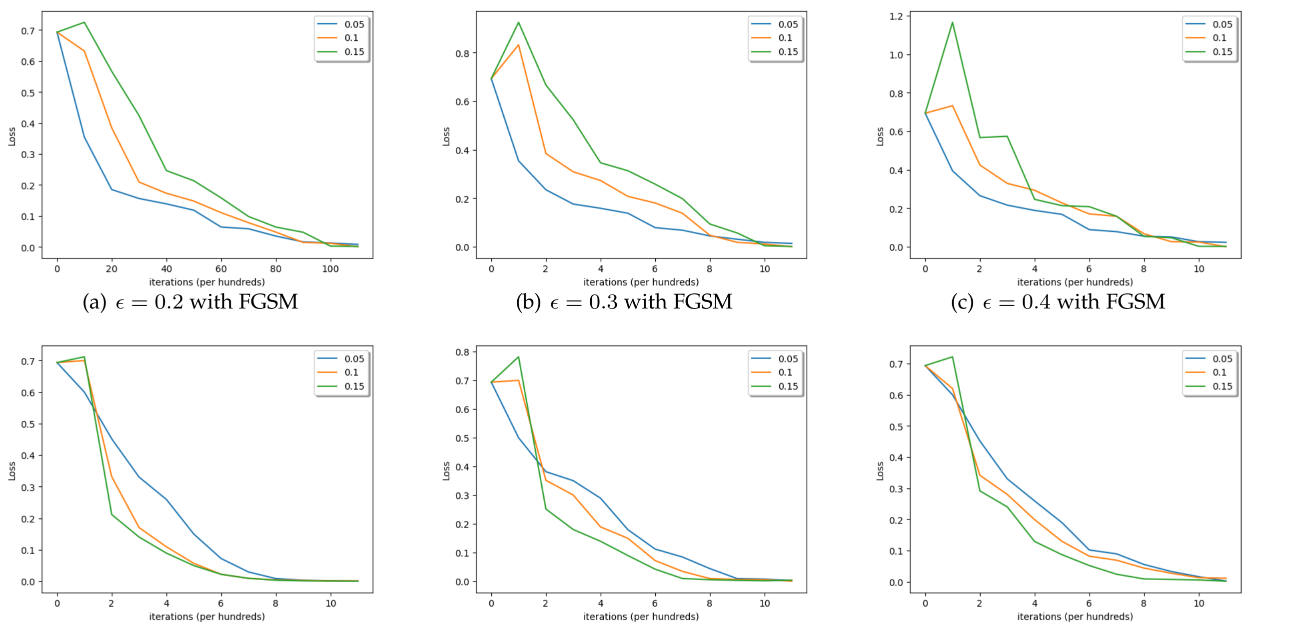
Experiment results. We first consider a one-dimensional adversarial training problem by using synthetic data, where the action set $\Omega_1 = \{x_1 \in \mathbb{R} : x_{\min} \leq x_1 \leq x_{\max}\}$ is endowed with a unit square form, while set $\Omega_2 = \{x_2 \in \mathbb{R} : \|x_2\|^2 \leq r\}$ is endowed with a Euclidean sphere form. Assign $\alpha_k = \frac{2}{k+1}$ as the step size. Take $\phi_1 = (x_1 - x_{\min})\log(x_1 - x_{\min}) + (x_{\max} - x_1)\log(x_{\max} - x_1)$, $\phi_2 = -\sqrt{r^2 - \|x\|_2^2}$, $\varphi_1 = \frac{1}{2}\|\sigma_1\|^2$ and $\varphi_2 = \frac{1}{2}\|\sigma_2\|^2$. The plot

$$\begin{aligned}\Gamma_1(x_1,\sigma_1,x_2)=&\frac{1}{q}\sum_{l=1}^{q}\sigma_{1,l}(-a_1x_1^TA_lx_1-2a_1x_1^TA_lx_2-a_1x_2^TA_lx_2-a_2\beta_1^Tx_1-a_2\beta_1^Tx_2)-\sigma_{1,l}\log(\frac{\sigma_{1,l}}{1-\sigma_{1,l}})-\log(1-\sigma_{1,l})\\&+(1-\beta_2)(a_1x_1^TA_lx_1+2a_1x_1^TA_lx_2+a_1x_2^TA_lx_2+a_2\beta_1^Tx_1+a_2\beta_{1,l}^Tx_2)+\frac{\lambda_1}{2}\|x_1\|^2-\frac{\lambda_2}{2}\|x_2\|^2.\end{aligned} \tag{17}$$



(a) $\epsilon = 0.2$ with FGSM
(b) $\epsilon = 0.3$ with FGSM
(c) $\epsilon = 0.4$ with FGSM
(d) $\epsilon = 0.2$ with PGD
(e) $\epsilon = 0.3$ with PGD
(f) $\epsilon = 0.4$ with PGD

Fig. 3. Trajectories of $Loss$ under MNIST dataset with different learning rates.



(a) $\epsilon = 0.2$ with FGSM
(b) $\epsilon = 0.3$ with FGSM
(c) $\epsilon = 0.4$ with FGSM
(d) $\epsilon = 0.2$ with PGD
(e) $\epsilon = 0.3$ with PGD
(f) $\epsilon = 0.4$ with PGD

Fig. 4. Trajectories of $Loss$ under the dataset for identifying cats with different learning rates.

of two players' payoffs is shown in Fig. 2. It can be seen that this non-convex game setting has a Nash stationary point and a global NE. Here we show $\boldsymbol{x}^{\dagger} = [-1.69, 1.90]$ as the Nash stationary point, while $\boldsymbol{x}^{\diamondsuit} = [1.63, 0.86]$ as the global NE. We compare Algorithm 1 with several well-known methods based on stationary information, such as the classic gradient descent (GD), the optimistic mirror descent (OMD) [41], and the extra-gradient method (EG) [73]. We randomly initialize these methods and the results are shown in Fig. 2 (a)-(f). Interestingly, the initialization has little impact on the convergence of our algorithm. However, it changes drastically the convergence property of other methods. The evolution results of $\boldsymbol{x}(t)$ with initial value $\boldsymbol{x}^0 = [2, -2.5]$ and $\boldsymbol{x}^0 = [3.5, 4.25]$ are shown in Fig. 2 (a) and (b), respectively, where all methods find the global NE. With initial value $\boldsymbol{x}^0 = [-4, -4]$ and $\boldsymbol{x}^0 = [0, -3]$ in Fig. 2 (c) and (d), only Algorithm 1 still achieves the target, while other methods are stuck into a Nash stationary point instead. Moreover, the evolution of $\boldsymbol{x}(t)$ with $\boldsymbol{x}^0 = [-1.5, 3.5]$ and $\boldsymbol{x}^0 = [-4, -0.5]$ can be found in Fig. 2 (e) and (f). In this condition, though these initial values are close to the Nash stationary point $\boldsymbol{x}^{\dagger}$, Algorithm 1 still works well and converges to the global NE.

TABLE 2
Test accuracy under attacks with different learning rates and perturbations.

	Learning rate	FGSM			PGD		
		$\epsilon = 0.2$	$\epsilon = 0.3$	$\epsilon = 0.4$	$\epsilon = 0.2$	$\epsilon = 0.3$	$\epsilon = 0.4$
[52]	0.05	92.51%	93.84%	94.72%	93.61%	92.47%	92.29%
	0.1	93.96%	92.21%	92.05%	94.11%	93.59%	93.20%
	0.15	93.18%	93.06%	92.58%	92.44%	92.31%	91.65%
[18]	0.05	90.88%	93.95%	92.48%	89.33%	91.75%	92.46%
	0.1	92.75%	92.33%	94.56%	92.25%	92.34%	93.87%
	0.15	96.24%	95.71%	94.05%	96.21%	95.15%	94.37%
[17]	0.05	96.21%	96.03%	95.88%	95.34%	94.71%	93.49%
	0.1	97.04%	96.66%	96.23%	96.00%	95.17%	94.22%
	0.15	96.79%	97.51%	96.82%	95.69%	95.50%	94.07%
Alg. 1 (ours)	0.05	**98.01**%	**97.66**%	**94.15**%	**97.12**%	**96.06**%	**93.81**%
	0.1	**98.52**%	**98.71**%	**96.22**%	**98.43**%	**97.12**%	**94.51**%
	0.15	**97.54**%	**96.33**%	**97.18**%	**97.05**%	**96.89**%	**94.24**%

Fig. 2 clearly shows that Algorithm 1 is an effective algorithm to seek the global NE regardless of the initial point, while other methods are susceptible to varied initial points.

Next, we examine Algorithm 1 on two real datasets: the MNIST dataset and the dataset for identifying cats. Consider two popular adversarial attacks: Fast Gradient Sign Method (FGSM) [74] and Projected Gradient Descent (PGD) [75]. In this view, we take different perturbations $\epsilon = 0.2, 0.3, 0.4$ on the original picture β_0, and show the trajectory with different learning rates $\alpha_k = 0.05, 0.1, 0.15$. Specifically, Figs. 3 and 4 correspond to the MNIST dataset and the dataset for identifying cats, respectively. In both Figs. 3 and 4, we observe that as the number of iterations increases, the value of $Loss$ decreases gradually, whose trend is dependent on different learning rates. Eventually, they all perform well and the $Loss$ tends to zero.

Moreover, based on the above settings, we compare Algorithm 1 with some state-of-the-art algorithms in robust neural network learning tasks [17], [18], [52]. Here, we focus on the MNIST dataset and consider perturbations $\epsilon = 0.2, 0.3, 0.4$ on the original dataset β_0. Table 2 shows the test accuracy of Algorithm 1 against two adversarial attacks FGSM and PGD, compared with other developed methods. The result indicates that our approach is effective under given perturbations, yielding a higher test accuracy than compared methods.

7.2 Sensor network localization

In this part, we apply our approach to solving a sensor network localization (SNL) problem.

Model and data pre-processing. We consider a class of non-convex games in SNL with N non-anchor sensor nodes and M anchor nodes [15], [51], [76]. The anchor nodes' positions are known while the non-anchor nodes' positions are unknown. For $i \in \mathcal{I}$, the position strategy set Ω_i is a unit square $\Omega_i = \{x_i \in \mathbb{R}^2 : x_{\min} \leq x_{il} \leq x_{\max}\}$ for $l = 1, 2$. The payoff function is defined as

$$\begin{aligned} &J_i(x_i, \boldsymbol{x}_{-i}) \qquad (20)\\ &= \sum_{j \in \mathcal{N}_s^i} (\|x_i - x_j\|^2 - d_{ij}^2)^2 + \sum_{k \in \mathcal{N}_a^i} (\|x_i - a_k\|^2 - e_{ik}^2)^2 + \frac{\kappa_x}{2}\|x_i\|^2. \end{aligned}$$

The first term in (20) is the same as (3), which measures the localization accuracy between non-anchor node i and its neighbor non-anchor node $j \in \mathcal{N}_s^i$. The second term in (20) is another localization measurement between non-anchor node i and its neighbor anchor node $k \in \mathcal{N}_a^i$. The last term serves as a regularizer. The goal is to estimate the position of all non-anchor sensor nodes as accurately as possible. Each non-anchor node i needs to satisfy $\|x_i^\diamondsuit - x_j^\diamondsuit\|^2 - d_{ij}^2 = 0$ and $\|x_i^\diamondsuit - a_k\|^2 - e_{ik}^2 = 0$ for any $j \in \mathcal{N}_s^i$ and $k \in \mathcal{N}_a^i$.

Analogously, we will conduct experiments on SNL problems using both artificial data and real data. Artificial data is generated to simulate scenarios with non-anchor nodes, intuitively demonstrating the convergence and effectiveness of the proposed algorithms. Then, we take two real datasets, the UJIIndoorLoc dataset and the hybrid indoor positioning dataset. The UJIIndoorLoc dataset was introduced in 2014 at the International Conference on Indoor Positioning and Indoor Navigation, which estimates user location based on building and floor. The hybrid indoor positioning dataset was created for the comparison and evaluation of positioning methods. Both datasets are available on the UC Irvine Machine Learning Repository website [77]. We extract the latitude and longitude coordinates of part of the sensors and standardize the data by doing min-max normalization.

Method. We reformulate this problem with a potential game model, where the potential function is

$$\begin{aligned} \Psi(\Lambda(x_i, \boldsymbol{x}_{-i})) = &\frac{1}{2}\sum_{i=1}^{N}\sum_{j \in \mathcal{N}_s^i} (\|x_i - x_j\|^2 - d_{ij})^2 \\ &+ \frac{1}{2}\sum_{i=1}^{N}\sum_{k \in \mathcal{N}_a^i} (\|x_i - a_k\|^2 - e_{ik}^2)^2 + \sum_{i=1}^{N}\frac{\kappa_x}{2}\|x_i\|^2. \end{aligned}$$

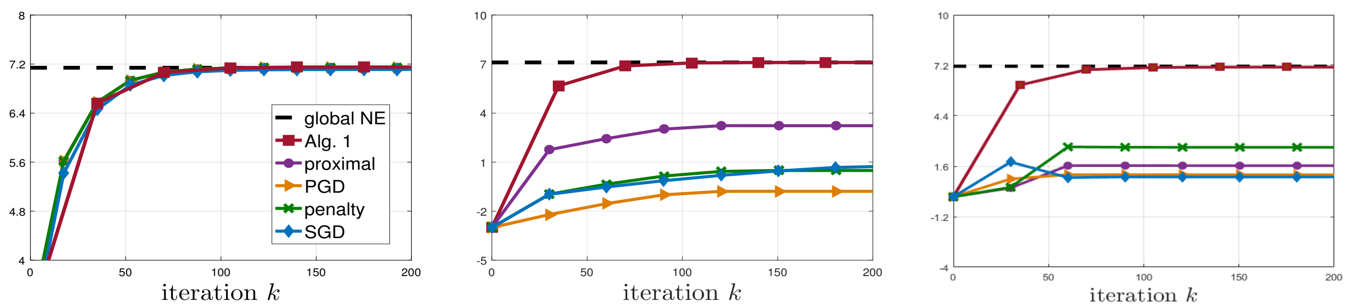
Then, we make a canonical transformation along the procedure in this paper to handle non-convexity. According to (9),

$$\mathscr{E}^+ = \Theta^* \cap \{\sigma : P(\sigma) + \kappa_x \boldsymbol{I}_{Nn} \succeq \kappa_x \boldsymbol{I}_{Nn}\}, \qquad (21)$$

which is a polyhedron here due to a common σ after canonical transformation. Moreover, it is worth noting that the global NE $\boldsymbol{x}^\diamondsuit$ represents the localization accuracy for all sensors. Deduced from the dual relation $\sigma^\diamondsuit =$

TABLE 3
MLE of five methods with three initialization and three iterations.

Initialization	Iteration	Values of MLE				
		Alg. 1 (ours)	proximal [45]	PGD [24]	penalty [25]	SGD [68]
$x_{11}^0 = 3$	5	**1.2631**	1.5589	1.0819	1.9259	1.6590
	50	**0.3009**	0.1460	0.1174	0.1625	0.2513
	200	**<u>0.0021</u>**	0.0343	0.0481	0.0567	0.0529
$x_{11}^0 = -3$	5	**9.4408**	10.0225	9.9922	9.9855	9.9925
	50	**3.3574**	4.8478	8.8478	7.5678	7.9568
	200	**<u>0.0379</u>**	3.8488	7.2688	6.5929	6.3470
$x_{11}^0 = -0.1$	5	**5.5911**	7.0914	6.8472	7.0160	5.6742
	50	**0.5977**	6.6912	6.0461	5.1330	5.8409
	200	**<u>0.0028</u>**	5.4908	5.1005	4.4875	6.1451



(a) $x_{11}(0) = 3$ (b) $x_{11}(0) = -3$ (c) $x_{11}(0) = -0.1$

Fig. 5. Comparison of convergence results with different initial points. Alg. 1 is ours.

$\nabla \Psi(\xi) \mid_{\xi=\Lambda(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)}$,

$$\sigma_{ij}^s = 2(\|x_i^\diamondsuit - x_j^\diamondsuit\|^2 - d_{ij}^2) = 0, \forall (i,j) \in \mathcal{E}_{ss},$$
$$\sigma_{ik}^a = 2(\|x_i^\diamondsuit - a_k\|^2 - e_{ik}^2) = 0, \forall (i,k) \in \mathcal{E}_{as},$$

where $\sigma = \text{col}\{\text{col}\{\sigma_{ij}^s\}_{(i,j)\in\mathcal{E}_{ss}}, \text{col}\{\sigma_{ik}^a\}_{(i,k)\in\mathcal{E}_{as}}\}$, $\mathcal{E}_{ss}$ is the set of edges between non-anchor nodes and $\mathcal{E}_{as}$ is the set of edges between anchor nodes and non-anchor nodes. These indicate that dual variable $\sigma^\diamondsuit$ corresponding to the global NE $\boldsymbol{x}^\diamondsuit$ is subject to $\sigma^\diamondsuit = \mathbf{0}_q$ where $q = |\mathcal{E}_{ss}| + |\mathcal{E}_{as}|$. As $\mathbf{0}_q \in \mathscr{E}^+$, we can replace $\mathscr{E}^+$ in (21) with a simple unit square constraint $\mathscr{E}^+ = [0, D]^q$ in the practical implementation, where D is a positive constant used to reduce the computational complexity.

On this basis, we employ Algorithm 1 to solve this problem. Assign $\alpha_k = \mathcal{O}(1/\sqrt{k})$ as the step size. Take $\phi_i(x_i) = \sum_{l=1}^{2}(x_{i,l} - x_{\min})\log(x_{i,l} - x_{\min}) + (x_{\max} - x_{i,l})\log(x_{\max} - x_{i,l})$ and $\varphi(\sigma) = \frac{1}{2}\|\sigma\|_2^2$. We will evaluate the performance by the *mean localization error*

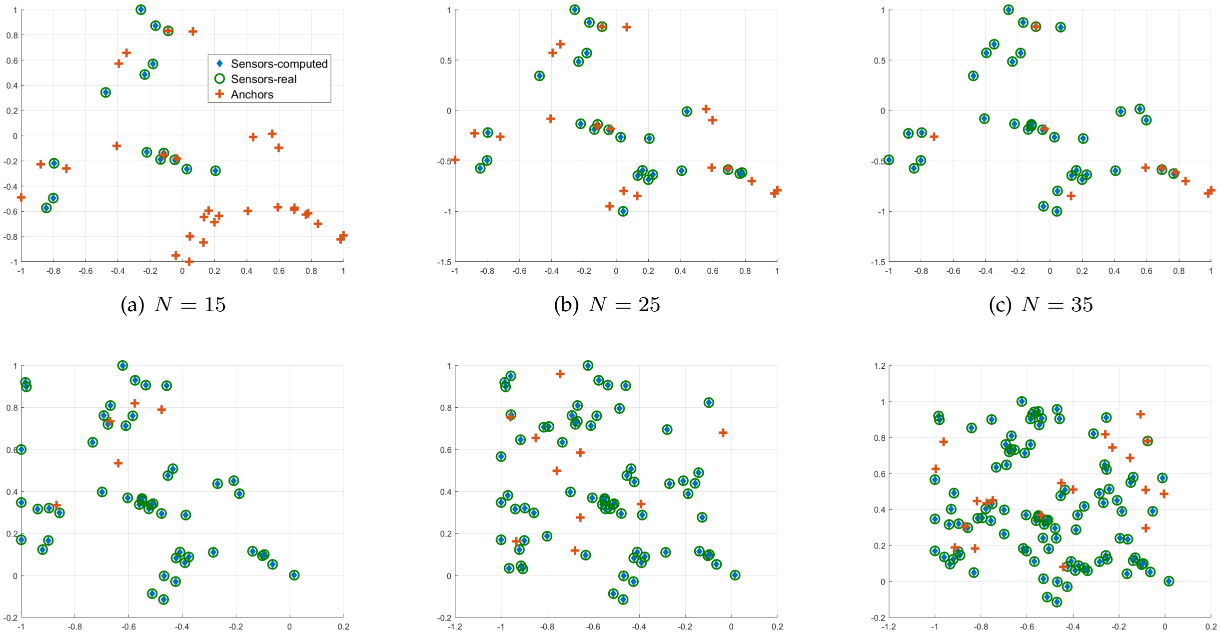
$$\text{MLE} = \frac{1}{N}\sqrt{\sum_{i=1}^{N}\|x_i - x_i^\diamondsuit\|^2}.$$

Set tolerance $t_{\text{tol}} = 10^{-3}$ and the terminal criterion $\|\boldsymbol{x}^{k+1} - \boldsymbol{x}^k\| \le t_{\text{tol}}$ and $\|\sigma^{k+1} - \sigma^k\| \le t_{\text{tol}}$.

To further illustrate this task, we will compare Algorithm 1 with several developed methods for multi-player models, including projected gradient descent (PGD) [24], penalty-based methods [25], stochastic gradient descent (SGD) [68], and gradient-proximal methods [45].

Experiment results We first consider a situation with artificial data where there are $N = 10$ non-anchor nodes and no anchor nodes. The distance parameters d_{ij} are randomly chosen from a compact region $[5, 10]$. We randomly generate three different initial points and record the value of MLE under 5, 50, and 200 iterations to measure the accuracy of the computed locations. We compare Algorithm 1 with several mentioned methods and list the results in Table 3. In the first case $x_{11}^0 = 3$, all methods locate the non-anchor nodes accurately, and the value of MLE decreases with the increase of iterations because the initial is near the global NE. Nevertheless, with two other initial conditions $x_{11}^0 = -3$ and $x_{11}^0 = -0.1$, the advantage of Algorithm 1 is outstanding, as shown in Table 3. Only Algorithm 1 maintains MLE within a tolerance error range, while other methods can not guarantee this. This is because Algorithm 1 is insusceptible to the initial point's location. In Fig. 5 (a)-(c), we further check these three cases in view of a fixed player's decision. It tells that only our algorithm achieves the target and finds the global NE, while others fail with varied initial points.

Next, via the two real datasets: the UJIIndoorLoc dataset and the hybrid indoor positioning dataset [77], we focus on the effectiveness of Algorithm 1 when the size of sensor networks expands. On the one hand, take different sensor numbers $N = 15, 25, 35, 50, 70, 100$ within the UJIIndoorLoc



(a) $N = 15$ (b) $N = 25$ (c) $N = 35$

(d) $N = 50$ (e) $N = 70$ (f) $N = 100$

Fig. 6. Performance of computed sensor location results with different network sizes.

TABLE 4
MLE of different methods with different sensor network sizes.

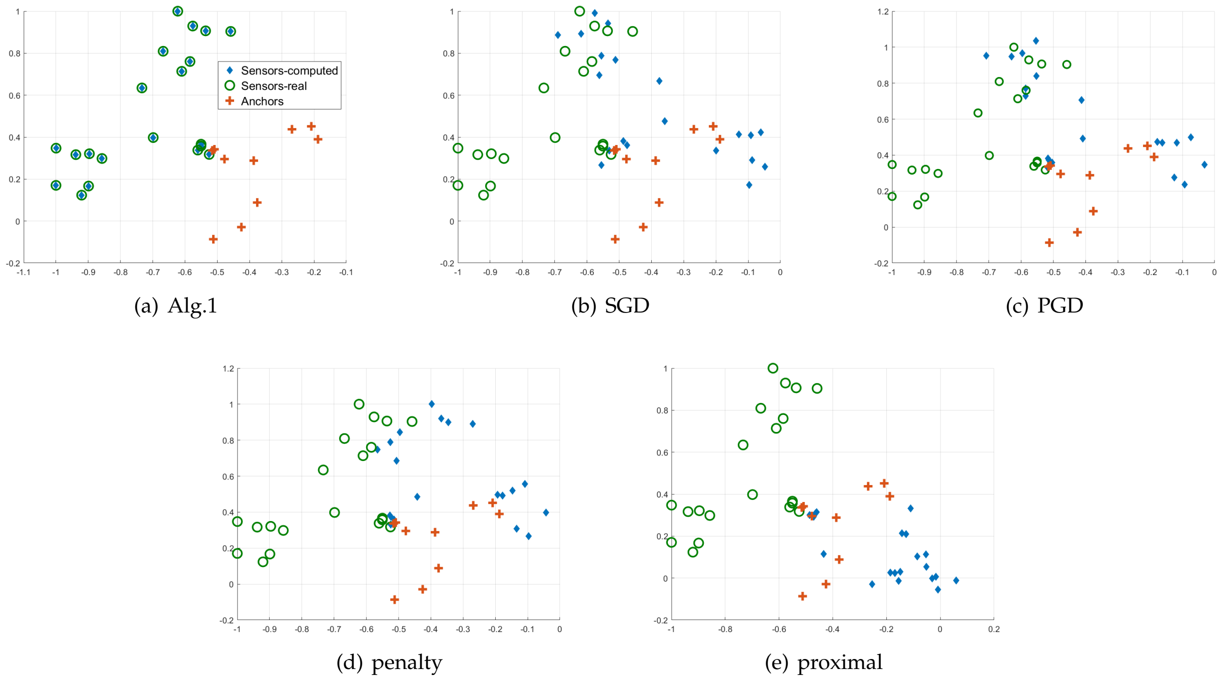
Node numbers	$N = 20$	$N = 30$	$N = 70$	$N = 100$
Alg. 1 (ours)	**0.0021**	**0.0008**	**0.0003**	**0.0002**
SGD	2.3396	2.2322	0.0811	0.3281
PGD	2.2700	2.2052	0.1891	0.2419
penalty	2.2696	2.7456	0.0817	0.3294
proximal	3.2578	2.3697	0.6794	0.4881

dataset and Fig. 6 shows the computed sensor location results in these cases. We can see that the good performance of Algorithm 1 is not influenced by network sizes. On the other hand, we compare Algorithm 1 with the developed algorithms under different sizes within the hybrid indoor positioning dataset. Fig. 7 shows the computed results from different methods for the case $N = 20$, which indicates that Algorithm 1 localizes all non-anchor nodes, while other methods exhibit some deviations from the true locations. Furthermore, we take $N = 20, 30, 70, 100$, and Table 4 lists the value of MLE in each case. It reflects that Algorithm 1 achieves a higher localization accuracy than the baseline methods. As the network size expands, only Algorithm 1 still maintains MLE in a tolerance range, while others cannot guarantee this.

8 Discussions

At last, we give some discussions on the obtained results in this paper, based on the comparisons with related works.

Firstly, we innovatively derived the existence condition of global NE in such a significant class of non-convex multi-player games. This game setting has broad applications in robust training [18], sensor localization [51], and mechanism design [20], [78]. Nevertheless, the obtained results cannot be achieved by other current approaches to non-convex game settings so far. With the rapid development and wide applications of adversarial systems and models in machine learning, non-convex games are playing a more and more important role in learning tasks. As we know, there have been some theoretical studies and algorithm designs for non-convex games, but most of these studies have focused on the two-player min-max problems. According to categories of players' objectives, the research status can be roughly divided into Polyak-Łojasiewicz cases [17], [26], strongly-concave cases [27], [28], and general non-convex non-concave cases [38], [39]. Such popularity is attributed to the success of GAN and its variants [40], [41]. Recently, learning methods in artificial intelligence have been developed for multi-agent systems, distributed design, federated learning, or cooperative estimation [78], [79], [80], [81]. This means that adversarial training is gradually generalized to multiple agents, no longer restricted to two opponents. There have been initial efforts for non-convex multi-player settings. For example, [42], [43] proposed a best-response scheme for local NE seeking, while [45] designed a gradient-proximal



(a) Alg.1 (b) SGD (c) PGD

(d) penalty (e) proximal

Fig. 7. Computed sensor location results with different methods

algorithm to find an approximate solution. As mentioned in these works, finding local NE or alternative approximations is challenging but acceptable. Up till now, revealing the existence of the global optimum or equilibrium in non-convex settings is still an open problem [21], [22]. In this view, it is important to investigate the existence condition of global NE in this paper. We employed conjugate transformation of duality theory and the continuous mapping of variational inequality to derive this in Theorem 1.

Secondly, we designed a continuous ODE to compute the global NE and deduced its discrete algorithm with its step-size designs and convergence rates in two typical cases. Recall the existing game-theoretical algorithms for multiple players based on the first-order information [23], [24], [25]. Most of these works depend on convexity assumptions, namely, strongly or strictly convex payoffs of each player or directly monotone pseudo-gradients, which are core to finding NE in game models. However, total convexity is a luxury in reality. Inevitably, the above approaches for convex game models perform unsatisfactorily when confronting non-convex settings, since their terminus merely lies in local NE or some approximations. Therefore, considering so important a class of non-convex in this paper, we are accountable for designing novel algorithms to approach the global NE. We realized the goal through the design of continuous conjugate-based ODE. This breaks the limits of traditional convexity assumptions in the study of multi-player game models. By using Lyapunov stability theory in nonlinear systems, we obtained the convergence of the designed dynamics in Theorem 2 and the exponential rate in Theorem 3, which demonstrate a strong convergence performance for continuous ODE. In addition, the induced discrete scheme also reached good convergence results in Theorem 4 and Theorem 5, respectively, for generalized monotone cases and potential cases.

Lastly, we give further discussions on the duality theory and associated techniques in this paper. Indeed, the applications of duality theory in machine learning are not rare [56], and the canonical duality theory utilized in this paper has also been studied before in optimization problems [14], [49]. Unfortunately, it is not straightforward to transplant this technique from optimization to game models. This is because players' decisions are coupled. When players make decisions, they also need to take into account the changes in other players' decision variables. This phenomenon yields that we should handle all players' decision variables as a unified profile. Therefore, we transformed the original problem into a complementary one by means of duality theory and assigned a sufficient feasible set for the dual variables. We finally overcame the above bottleneck by virtue of variational inequalities, and obtained the existence condition of the global NE. Besides, the convergence proof of the proposed algorithms to seek the global NE differs from those of optimization. Unlike optimization, where there is a uniform objective function for all variables, multi-player game models require new measures to analyze the convergence of realization algorithms. To this end, we utilized the Bregman divergence and some inequalities to overcome obstacles, including the three-point identity, Fenchel's inequality, and Jensen's inequality. Adopting these techniques in the theoretical analysis may provide a new path in the future study of large-scale multi-player interactions and interference.

9 Conclusions and Future Work

We considered a typical class of non-convex multi-player games and discussed how to approach the global NE. By virtue of canonical duality theory and VI problems, we proposed a conjugate-based ODE for the solution of a transformed VI problem. Thus, we can derive the global NE of the original non-convex game if the duality relation can be verified. After providing theoretical guarantees of the ODE convergence, we conducted discretization and analyzed step-size settings for the corresponding convergence rates under two typical non-convex conditions.

Our exploration continues to advance. Regarding convergence rates, combining proper accelerated methods may yield promising results; In terms of the multi-player background, players' interaction may rely on a communication network in consideration of privacy and security, which may suggest the necessity for a distributed or decentralized protocol.

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Supplementary Materials

In the Supplementary Materials, we provide several detailed sections that were omitted from the main text. The necessary preliminaries include canonical duality theory, variational inequalities, Bregman divergence, and several significant inequalities. Also, we provide detailed proofs for all lemmas and theorems.

S.1 Canonical duality theory

We begin the supplementary section of this paper with the following fundamental concepts of canonical duality theory. A differentiable function $\Psi : \Theta \to \mathbb{R}$ is said to be a canonical function if its derivative $\nabla\Psi : \Theta \to \Theta^*$ is a one-to-one mapping. Besides, if Ψ is a convex canonical function, its conjugate function $\Psi^* : \Theta^* \to \mathbb{R}$ can be uniquely defined by the Legendre transformation, that is,

$$\Psi^*(\sigma) = \left\{\xi^T\sigma - \Psi(\xi) \mid \sigma = \nabla\Psi(\xi)\right\},$$

where $\sigma \in \Theta^*$ is a canonical dual variable. On this basis, there are corresponding canonical duality relations holding on $\Theta \times \Theta^*$:

$$\begin{aligned} &\sigma = \nabla\Psi(\xi), \\ \Leftrightarrow \quad &\xi = \nabla\Psi^*(\sigma), \\ \Leftrightarrow \quad &\xi^T\sigma = \Psi(\xi) + \Psi^*(\sigma). \end{aligned}$$

Here, (ξ, σ) is called the Legendre canonical duality pair on $\Theta \times \Theta^*$.

S.2 Proof of Lemma 1

In this section, we give the proof of Lemma 1, which investigates the relationship of stationary points between (8) and (1). Here we reclaim Lemma 1 for convenience.

Lemma 1 For a profile $\boldsymbol{x}^\diamondsuit$, if there exists $\boldsymbol{\sigma}^\diamondsuit \in \boldsymbol{\Theta}^*$ such that for $i \in \mathcal{I}$, $(x_i^\diamondsuit, \sigma_i^\diamondsuit)$ is a stationary point of complementarity function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}^\diamondsuit)$, then $\boldsymbol{x}^\diamondsuit$ is a Nash stationary point of game (1).

Proof. For a given strategy profile $\boldsymbol{x}^\diamondsuit$, if there exists $\boldsymbol{\sigma}^\diamondsuit \in \boldsymbol{\Theta}^*$ such that for all $i \in \mathcal{I}$, $(x_i^\diamondsuit, \sigma_i^\diamondsuit)$ is a stationary point of complementarity function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}^\diamondsuit)$, then it satisfies the following first-order conditions:

$$\mathbf{0}_n \in \sigma_i^{\diamondsuit T}\nabla_{x_i}\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \mathcal{N}_{\Omega_i}(x_i^\diamondsuit), \tag{s.1a}$$

$$\mathbf{0}_{q_i} \in -\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \nabla\Psi_i^*(\sigma_i^\diamondsuit) + \mathcal{N}_{\Theta_i^*}(\sigma_i^\diamondsuit), \tag{s.1b}$$

where $\mathcal{N}_{\Omega_i}(x_i^\diamondsuit)$ is the normal cone at point $x_i^\diamondsuit$ on set Ω_i, with a similar definition for the normal cone $\mathcal{N}_{\Theta_i^*}(\sigma_i^\diamondsuit)$. Following the definition of the convex canonical function Ψ_i, we can learn that its derivative $\nabla\Psi_i : \Theta_i \to \Theta_i^*$ is a one-to-one mapping from Θ_i to its range Θ_i^*. Thus, for given $\xi_i^\diamondsuit \in \Theta_i$ with $\xi_i^\diamondsuit = \Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)$, there exists a unique $\sigma_i^\diamondsuit \in \Theta_i^*$ such that

$$\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i^\diamondsuit).$$

Meanwhile, given this Legendre canonical duality pair $(\xi_i^\diamondsuit, \sigma_i^\diamondsuit)$ on $\Theta_i \times \Theta_i^*$, the duality relation holds that

$$\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i^\diamondsuit) \iff \xi_i^\diamondsuit = \nabla\Psi_i^*(\sigma_i^\diamondsuit).$$

With all this in mind, (s.1b) can be transformed into

$$\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) = \nabla\Psi_i^*(\sigma_i^\diamondsuit). \tag{s.2}$$

Using the duality relation again, (s.2) is equivalent to

$$\sigma_i^\diamondsuit = \nabla\Psi_i(\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)). \tag{s.3}$$

By substituting (s.3) into (s.1a), we have

$$\mathbf{0}_n \in \nabla\Psi_i(\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit))^T\nabla_{x_i}\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \mathcal{N}_{\Omega_i}(x_i^\diamondsuit). \tag{s.4}$$

According to the chain rule,

$$\nabla\Psi_i(\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit))^T\nabla_{x_i}\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) = \nabla_{x_i}J_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit).$$

Therefore, (s.4) is equivalent to

$$\mathbf{0}_n \in \nabla_{x_i}J_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \mathcal{N}_{\Omega_i}(x_i^\diamondsuit). \tag{s.5}$$

Since (s.5) is true for any player $i \in \mathcal{I}$, the profile $\boldsymbol{x}^\diamondsuit$ satisfies the Nash stationary condition, which completes the proof. □

S.3 VARIATIONAL INEQUALITY

In this section, we introduce some concepts and properties related to the variational inequality (VI). Recall the following notations according to problem (8)

$$\boldsymbol{z} = \operatorname{col}\{\boldsymbol{x}, \boldsymbol{\sigma}\}, \quad \boldsymbol{\Xi} = \boldsymbol{\Omega} \times \mathscr{E}^+ \subset \mathbb{R}^{nN+q}.$$

For the conjugate gradient of canonical function Ψ_i for $i \in \mathcal{I}$, denote

$$\nabla\Psi^*(\boldsymbol{\sigma}) = \operatorname{col}\{\nabla\Psi_i^*(\sigma_i)\}_{i=1}^N.$$

Also, denote the profile of all Λ_i by

$$\Lambda(\boldsymbol{x}) = \operatorname{col}\{\Lambda_i(x_i, \boldsymbol{x}_{-i})\}_{i=1}^N,$$

and the augmented partial derivative profile as follows

$$G(\boldsymbol{x}, \boldsymbol{\sigma}) = \operatorname{col}\left\{\sigma_i^T \nabla_{x_i}\Lambda_i(x_i, \boldsymbol{x}_{-i})\right\}_{i=1}^N.$$

In this way, the pseudo-gradient of (8) can be rewritten as

$$F(\boldsymbol{z}) \triangleq \begin{bmatrix} G(\boldsymbol{x}, \boldsymbol{\sigma}) \\ -\Lambda(\boldsymbol{x}) + \nabla\Psi^*(\boldsymbol{\sigma}) \end{bmatrix} = \begin{bmatrix} \operatorname{col}\{\sum_{k=1}^{q_i}[\sigma_i]_k \nabla_{x_i}\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i})\}_{i=1}^N \\ \operatorname{col}\{-\Lambda_i(x_i, \boldsymbol{x}_{-i}) + \nabla\Psi_i^*(\sigma_i)\}_{i=1}^N \end{bmatrix}. \tag{s.6}$$

To proceed, the corresponding variational inequality (VI) problem $\mathrm{VI}(\boldsymbol{\Xi}, F)$ is defined as

$$\text{Find} \quad \boldsymbol{z} \in \boldsymbol{\Xi} \quad \text{subject to} \quad (\boldsymbol{z}' - \boldsymbol{z})^T F(\boldsymbol{z}) \geq 0, \quad \forall \boldsymbol{z}' \in \boldsymbol{\Xi}. \tag{s.7}$$

The solution to this VI problem is denoted by $\mathrm{SOL}(\boldsymbol{\Xi}, F)$. Moreover, since $F(\boldsymbol{z})$ is a continuous mapping and $\boldsymbol{\Xi}$ is a closed set, we have the following result, referring to [30, Page 2-3].

Lemma S1. The solution set $\mathrm{SOL}(\boldsymbol{\Xi}, F)$ of $\mathrm{VI}(\boldsymbol{\Xi}, F)$ in (s.7) is closed. Moreover, any profile $\boldsymbol{z} \in \mathrm{SOL}(\boldsymbol{\Xi}, F)$ if and only if

$$\mathbf{0}_{nN+q} \in F(\boldsymbol{z}) + \mathcal{N}_{\boldsymbol{\Xi}}(\boldsymbol{z}).$$

S.4 PROOF OF THEOREM 1

In this section, we give the proof of Theorem 1 based on the aforementioned preparation regarding the canonical duality theory and properties in VI problems. For convenience, we reproduce Theorem 1 here.

Theorem 1 There exists $\boldsymbol{x}^\diamondsuit$ as the global NE of the non-convex multi-player game (1) if $(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ is a solution to $\mathrm{VI}(\boldsymbol{\Xi}, F)$ with $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i)\mid_{\xi_i = \Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)}$ for $i \in \mathcal{I}$.

Proof. If there exists $\boldsymbol{\sigma}^\diamondsuit \in \mathscr{E}^+$ such that $\boldsymbol{z}^\diamondsuit = \operatorname{col}\{\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit\}$ is a solution to $\mathrm{VI}(\boldsymbol{\Xi}, F)$, then it follows from Lemma S1 that

$$\mathbf{0}_{nN+q} \in F(\boldsymbol{z}^\diamondsuit) + \mathcal{N}_{\boldsymbol{\Xi}}(\boldsymbol{z}^\diamondsuit). \tag{s.8}$$

This implies that for $i \in \mathcal{I}$,

$$\begin{aligned} \mathbf{0}_n &\in \sigma_i^{\diamondsuit T} \nabla_{x_i}\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \mathcal{N}_{\Omega_i}(x_i^\diamondsuit), \\ \mathbf{0}_{q_i} &\in -\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \nabla\Psi_i^*(\sigma_i^\diamondsuit) + \mathcal{N}_{\mathscr{E}_i^+}(\sigma_i^\diamondsuit), \end{aligned}$$

or equivalently described as

$$\begin{aligned} (\sigma_i^{\diamondsuit T} \nabla_{x_i}\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit))^T (x_i - x_i^\diamondsuit) \geq 0, &\quad \forall x_i \in \Omega_i, \\ (-\Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) + \nabla\Psi_i^*(\sigma_i^\diamondsuit))^T (\sigma_i - \sigma_i^\diamondsuit) \geq 0, &\quad \forall \sigma_i \in \mathscr{E}_i^+. \end{aligned} \tag{s.9}$$

Moreover, if $\sigma_i^\diamondsuit = \nabla\Psi_i(\xi_i)\mid_{\xi_i = \Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)}$ for $i \in \mathcal{I}$, then the canonical duality relation hold on $\Theta_i \times \mathscr{E}_i^+$ for $i \in \mathcal{I}$. This indicates that the solution to $\mathrm{VI}(\boldsymbol{\Xi}, F)$ is a stationary point profile of (8) on $\Theta_i \times \Theta_i^*$.

Thus, similar to the chain rules used in Lemma 1, we can further derive that

$$(\nabla_{x_i} J_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit))^T (x_i - x_i^\diamondsuit) \geq 0, \quad \forall x_i \in \Omega_i.$$

Moreover, when $\sigma_i \in \mathscr{E}_i^+$, the Hessian matrix satisfies

$$\nabla_{x_i}^2 \Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i}) = \sum_{k=1}^{q_i} [\sigma_i]_k \nabla_{x_i}^2 \Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) \succeq \kappa_x \boldsymbol{I}_n,$$

which indicates the convexity of $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ with respect to x_i. Besides, due to the convexity of Ψ_i, its Legendre conjugate Ψ_i^* is also convex [82]. Therefore, the total complementary function $\Gamma_i(x_i, \sigma_i, \boldsymbol{x}_{-i})$ is concave in canonical dual variable σ_i.

In this light, we can establish the global optimality of $(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ on $\boldsymbol{\Omega} \times \mathscr{E}^+$, i.e., for $i \in \mathcal{I}$,

$$\Gamma_i(x_i^\diamondsuit, \sigma_i, \boldsymbol{x}_{-i}^\diamondsuit) \leq \Gamma_i(x_i^\diamondsuit, \sigma_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) \leq \Gamma_i(x_i, \sigma_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit), \quad \forall x_i \in \Omega_i, \ \sigma_i \in \mathscr{E}_i^+.$$

The inequality relation above indicates that

$$J_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit) \le J_i(x_i, \boldsymbol{x}_{-i}^\diamondsuit), \quad \forall x_i \in \Omega_i, \quad \forall i \in \mathcal{I}.$$

This confirms that $\boldsymbol{x}^\diamondsuit$ is the global NE of (1), which completes the proof. □

S.5 Convergence analysis of the conjugate-based ODE

In this section, we aim to provide proofs for Lemma 2, Theorems 2 and 3. We first introduce some preliminaries that are necessary for the convergence analysis of ODE (13), which are also widely-accepted concepts in convex analysis. Take $h(z) : \Xi \to \mathbb{R}$ as a differentiable ω-strongly convex function on a closed convex set Ξ, which satisfies

$$h\left(\theta z + (1-\omega)z'\right) \le \theta h(z) + (1-\theta) h\left(z'\right) - \frac{\omega}{2}\theta(1-\theta)\left\|z'-z\right\|^2, \quad \forall z, z' \in \Xi,\ \theta \in [0,1].$$

Additionally, h admits a Lipschitz continuous gradient if there exists a constant $L > 0$, such that

$$\|\nabla h(z') - \nabla h(z)\| \le L\|z - z'\|, \quad \forall z, z' \in \Xi,$$

which is equivalent to

$$h\left(z'\right) - h(z) \le \left(z'-z\right)^{\mathrm{T}} \nabla h(z) + \frac{L}{2}\left\|z - z'\right\|^2, \quad \forall z, z' \in \Xi.$$

On the other hand, according to the duality theory [60], the conjugate function of h defined on the dual space Ξ^* is given by

$$h^*(s) = \sup_{z\in\Xi}\left\{z^T s - h(z)\right\},$$

where $s \in \Xi^*$ serves as a dual variable. Additionally, if we consider h as a differentiable and strongly convex function on a closed convex set Ξ, then according to [31], h^* is also convex and differentiable on Ξ^*, and satisfies

$$h^*(s) = \min_{z\in\Xi}\left\{-z^T s + h(z)\right\}.$$

Moreover, the conjugate gradient $\nabla h^*(s)$ who maps Ξ^* to Ξ satisfies

$$\nabla h^*\left(s\right) = \operatorname{argmin}_{z\in\Xi}\left\{-z^T s + h(z)\right\}.$$

With these preliminaries in hand, we can investigate the convergence of the ODE (13). For simplicity, let us denote the following compact forms associated with the gradients therein

$$\nabla\boldsymbol{\phi}(\boldsymbol{x}) \triangleq \operatorname{col}\left\{\nabla\phi_i\left(x_i\right)\right\}_{i=1}^N, \quad \nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) = \operatorname{col}\{\nabla\varphi_i(\sigma_i)\}_{i=1}^N;$$

$$\nabla\boldsymbol{\phi}^*(\boldsymbol{y}) = \operatorname{col}\{\nabla\phi_i^*\left(y_i\right)\}_{i=1}^N, \quad \nabla\boldsymbol{\varphi}^*(\boldsymbol{\nu}) = \operatorname{col}\{\nabla\varphi_i^*\left(\nu_i\right)\}_{i=1}^N.$$

Hence, together with the compact forms $G(\boldsymbol{x}, \boldsymbol{\sigma})$, $\Lambda\left(\boldsymbol{x}\right)$, and $\nabla\Psi^*\left(\boldsymbol{\sigma}\right)$ defined in (s.6), ODE (13) can be compactly presented as

$$\begin{cases} \dot{\boldsymbol{y}} = -G(\boldsymbol{x}, \boldsymbol{\sigma}) + \nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}, \\ \dot{\boldsymbol{\nu}} = \Lambda\left(\boldsymbol{x}\right) - \nabla\Psi^*\left(\boldsymbol{\sigma}\right) + \nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}, \\ \boldsymbol{x} = \nabla\boldsymbol{\phi}^*(\boldsymbol{y}), \\ \boldsymbol{\sigma} = \nabla\boldsymbol{\varphi}^*\left(\boldsymbol{\nu}\right). \end{cases} \tag{s.10}$$

On this basis, we first show a relationship between the equilibrium in ODE (s.10) (or ODE (13)) and the global NE of game (1). Rewrite Lemma 2 here for convenience.

Lemma 2 Suppose that $(\boldsymbol{y}^\diamondsuit, \boldsymbol{\nu}^\diamondsuit, \boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ is an equilibrium point of ODE (13). If $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\xi_i\right)|_{\xi_i = \Lambda_i\left(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i \in \mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of (1).

Proof. If $(\boldsymbol{y}^\diamondsuit, \boldsymbol{x}^\diamondsuit, \boldsymbol{\nu}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)$ is an equilibrium point of ODE (s.10), we have

$$\mathbf{0}_{nN} = -G(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit) + \nabla\boldsymbol{\phi}(\boldsymbol{x}^\diamondsuit) - \boldsymbol{y}^\diamondsuit, \tag{s.11a}$$

$$\mathbf{0}_q = \Lambda\left(\boldsymbol{x}^\diamondsuit\right) - \nabla\Psi^*\left(\boldsymbol{\sigma}^\diamondsuit\right) + \nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}^\diamondsuit) - \boldsymbol{\nu}^\diamondsuit, \tag{s.11b}$$

$$\boldsymbol{x}^\diamondsuit = \nabla\boldsymbol{\phi}^*(\boldsymbol{y}^\diamondsuit), \tag{s.11c}$$

$$\boldsymbol{\sigma}^\diamondsuit = \nabla\boldsymbol{\varphi}^*(\boldsymbol{\nu}^\diamondsuit). \tag{s.11d}$$

It follows from $\boldsymbol{y}^\diamondsuit = -G(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit) + \nabla\boldsymbol{\phi}(\boldsymbol{x}^\diamondsuit)$ that (s.11c) becomes

$$\boldsymbol{x}^\diamondsuit = \nabla\boldsymbol{\phi}^*(-G(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit) + \nabla\boldsymbol{\phi}(\boldsymbol{x}^\diamondsuit)). \tag{s.12}$$

For $i \in \mathcal{I}$, (s.12) is equivalent to

$$x_i = \nabla\phi_i^*(-\sigma_i^T \nabla_{x_i}\Lambda_i\left(x_i, \boldsymbol{x}_{-i}\right) + \nabla\phi_i(x_i)).$$

Moreover, by recalling

$$\nabla\phi_i^*(y_i) = \mathrm{argmin}_{x_i\in\Omega_i}\left\{-x_i^T y_i + \phi_i(x_i)\right\},$$

and taking y_i as $-\sigma_i^T\nabla_{x_i}\Lambda_i\left(x_i,\boldsymbol{x}_{-i}\right)+\nabla\phi_i(x_i)$, we can obtain the associated first-order condition, expressed in the following compact form

$$\mathbf{0}_{nN} \in G(\boldsymbol{x}^\diamondsuit,\boldsymbol{\sigma}^\diamondsuit) + \mathcal{N}_{\boldsymbol{\Omega}}(\boldsymbol{x}^\diamondsuit). \tag{s.13}$$

Similarly, it follows from (s.11b) and (s.11d) that

$$\boldsymbol{\sigma}^\diamondsuit = \nabla\boldsymbol{\varphi}^*(\Lambda\left(\boldsymbol{x}^\diamondsuit\right) - \nabla\Psi^*\left(\boldsymbol{\sigma}^\diamondsuit\right) + \nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}^\diamondsuit)),$$

which yields

$$\mathbf{0}_q \in -\Lambda\left(\boldsymbol{x}^\diamondsuit\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}^\diamondsuit\right) + \mathcal{N}_{\mathscr{E}^+}(\boldsymbol{\sigma}^\diamondsuit). \tag{s.14}$$

Thus, by combining (s.13) and (s.14), it follows from Lemma S1 that $\boldsymbol{z}^\diamondsuit = \mathrm{col}\{\boldsymbol{x}^\diamondsuit,\boldsymbol{\sigma}^\diamondsuit\}$ is a solution to VI($\boldsymbol{\Xi}, F$). Moreover, according to Theorem 1, the solution of VI($\boldsymbol{\Xi}, F$) derives a global NE of game (1) if $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit,\boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i\in\mathcal{I}$, which completes the proof. $\square$

Now, we are in a position to prove the convergence of conjugate-based ODE (13). Let us recall Theorem 2 for convenience.

Theorem 2 If $\mathscr{E}_i^+$ is nonempty for $i\in\mathcal{I}$, then ODE (13) is bounded and convergent. Moreover, if the convergent point $(\boldsymbol{y}^\diamondsuit,\boldsymbol{\nu}^\diamondsuit,\boldsymbol{x}^\diamondsuit,\boldsymbol{\sigma}^\diamondsuit)$ satisfies $\sigma_i^\diamondsuit = \nabla\Psi_i\left(\xi_i\right)|_{\xi_i=\Lambda_i\left(x_i^\diamondsuit,\boldsymbol{x}_{-i}^\diamondsuit\right)}$ for $i\in\mathcal{I}$, then $\boldsymbol{x}^\diamondsuit$ is the global NE of (1).

Proof. (i) We first prove that the trajectory ($\boldsymbol{y}(t)$, $\boldsymbol{x}(t)$, $\boldsymbol{\nu}(t)$, $\boldsymbol{\sigma}(t)$) is bounded along ODE (13). Construct a Lyapunov candidate function as

$$V_1 = \sum_{i=1}^N D_{\phi_i^*}(y_i,y_i^\diamondsuit) + D_{\varphi_i^*}(\nu_i,\nu_i^\diamondsuit). \tag{s.15}$$

Here, the Bregman divergences are expressed in detail as follows:

$$\begin{aligned}D_{\phi_i^*}(y_i,y_i^\diamondsuit) &= \phi_i^*(y_i) - \phi_i^*(y_i^\diamondsuit) - \nabla\phi_i^*(y_i^\diamondsuit)^T(y_i - y_i^\diamondsuit),\\ D_{\varphi_i^*}(\nu_i,\nu_i^\diamondsuit) &= \varphi_i^*(\nu_i) - \varphi_i^*(\nu_i^\diamondsuit) - \nabla\varphi_i^*(\nu_i^\diamondsuit)^T(\nu_i - \nu_i^\diamondsuit).\end{aligned}$$

Consider the term $D_{\phi_i^*}(y_i,y_i^\diamondsuit)$ for $i\in\mathcal{I}$. Since $x_i = \nabla\phi_i^*\left(y_i\right)$ and $x_i^\diamondsuit = \nabla\phi_i^*\left(y_i^\diamondsuit\right)$, it follows from the expression of $\nabla\phi_i^*$ in (11) that

$$\phi_i^*(y_i) = x_i^T y_i - \phi_i(x_i), \quad \phi_i^*(y_i^\diamondsuit) = x_i^{\diamondsuit T} y_i^\diamondsuit - \phi_i(x_i^\diamondsuit). \tag{s.16}$$

Thus, by (s.16), we get

$$\begin{aligned}D_{\phi_i^*}(y_i,y_i^\diamondsuit) &= \phi_i^*(y_i) - \phi_i^*(y_i^\diamondsuit) - \nabla\phi_i^*(y_i^\diamondsuit)^T(y_i - y_i^\diamondsuit)\\ &= \phi_i(x_i^\diamondsuit) - \phi_i(x_i) - (x_i^\diamondsuit - x_i)^T y_i\\ &= \phi_i(x_i^\diamondsuit) - \phi_i(x_i) - (x_i^\diamondsuit - x_i)^T\nabla\phi(x_i) + (x_i^\diamondsuit - x_i)^T\nabla\phi(x_i) - (x_i^\diamondsuit - x_i)^T y_i.\end{aligned}$$

Since ϕ_i is μ_x-strongly convex on Ω_i, we can further derive

$$D_{\phi_i^*}(y_i,y_i^\diamondsuit) \geq \frac{\mu_x}{2}\left\|x_i - x_i^\diamondsuit\right\|^2 + (x_i^\diamondsuit - x_i)^T(\nabla\phi(x_i) - y_i),$$

which yields

$$\sum_{i=1}^N D_{\phi_i^*}(y_i,y_i^\diamondsuit) \geq \frac{\mu_x}{2}\left\|\boldsymbol{x} - \boldsymbol{x}^\diamondsuit\right\|^2 + \sum_{i=1}^N (x_i^\diamondsuit - x_i)^T\left(\nabla\phi_i\left(x_i\right) - y_i\right). \tag{s.17}$$

In fact, recall $\nabla\phi_i^*\left(y_i\right) = \mathrm{argmin}_{x\in\Omega_i}\left\{-x^T y_i + \phi_i(x)\right\}$. Due to the optimality of $\nabla\phi_i^*\left(y_i\right)$ and the convexity of ϕ_i, we have

$$\left(\nabla\phi_i^*(y_i)\right)^T\left(\nabla\phi_i\left(\nabla\phi_i^*\left(y_i\right)\right) - y_i\right) \leq \left(\nabla\phi_i^*(y_i^\diamondsuit)\right)^T\left(\nabla\phi_i\left(\nabla\phi_i^*\left(y_i\right)\right) - y_i\right). \tag{s.18}$$

Furthermore, in consideration of $x_i = \nabla\phi_i^*\left(y_i\right)$ and $x_i^\diamondsuit = \nabla\phi_i^*(y_i^\diamondsuit)$ again, (s.18) implies

$$\begin{aligned}0 &\leq \nabla\phi_i^*\left(y_i^\diamondsuit\right)^T\left(\nabla\phi_i\left(\nabla\phi_i^*\left(y_i\right)\right) - y_i\right) - \nabla\phi_i^*\left(y_i\right)^T\left(\nabla\phi_i\left(\nabla\phi_i^*\left(y_i\right)\right) - y_i\right)\\ &= (x_i^\diamondsuit - x_i)^T\left(\nabla\phi_i\left(x_i\right) - y_i\right).\end{aligned} \tag{s.19}$$

Thus, (s.17) becomes

$$\sum_{i=1}^N D_{\phi_i^*}(y_i,y_i^\diamondsuit) \geq \frac{\mu_x}{2}\left\|\boldsymbol{x} - \boldsymbol{x}^\diamondsuit\right\|^2.$$

The analogous analysis of the term $D_{\varphi_i^*}(\nu_i, \nu_i^\diamondsuit)$ in (s.15) can be conducted, revealing that

$$\sum_{i=1}^{N} D_{\varphi_i^*}(\nu_i, \nu_i^\diamondsuit) \geq \frac{\mu_\sigma}{2} \left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\right\|^2 + \sum_{i=1}^{N} (\sigma_i^\diamondsuit - \sigma_i)^T \left(\nabla\varphi_i\left(\sigma_i\right) - \nu_i\right).$$

Besides, recalling $\nabla\varphi_i^*\left(\nu_i\right) = \operatorname{argmin}_{\sigma_i \in \mathscr{E}_i^+}\{-\sigma_i^T \nu_i + \varphi_i(\sigma_i)\}$, and leveraging the convexity of φ_i along with the optimality of $\nabla\varphi_i^*\left(\nu_i\right)$, we deduce

$$0 \leq (\sigma_i^\diamondsuit - \sigma_i)^T \left(\nabla\varphi_i\left(\sigma_i\right) - \nu_i\right). \tag{s.20}$$

This similarly leads to

$$\sum_{i=1}^{N} D_{\varphi_i^*}(\nu_i, \nu_i^\diamondsuit) \geq \frac{\mu_\sigma}{2} \left\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\right\|^2.$$

As a result, we obtain the lower bound of (s.15) as below

$$V_1 \geq \mu(\|\boldsymbol{x} - \boldsymbol{x}^\diamondsuit\|^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\|^2) \geq 0,$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$. This means that V_1 is positive semi-definite, and $V_1 = 0$ if and only if $\boldsymbol{x} = \boldsymbol{x}^\diamondsuit$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\diamondsuit$. Moreover, V_1 is radially unbounded in $\boldsymbol{x}(t)$ and $\boldsymbol{\sigma}(t)$.

Next, we investigate the derivative of V_1 along ODE (13), that is,

$$\begin{aligned}
\frac{d}{dt}V_1(t) &= \frac{d}{dt}\sum_{i=1}^{N} D_{\phi_i^*}(y_i, y_i^\diamondsuit) + D_{\varphi_i^*}(\nu_i, \nu_i^\diamondsuit) \\
&= \frac{d}{dt}\sum_{i=1}^{N} (\phi_i^*(y_i) - \phi_i^*(y_i^\diamondsuit) - (y_i - y_i^\diamondsuit)^T \nabla\phi_i^*\left(y_i^*\right)) + \frac{d}{dt}\sum_{i=1}^{N} (\varphi_i^*(\nu_i) - \varphi_i^*(\nu_i^\diamondsuit) - (\nu_i - \nu_i^\diamondsuit)^T \nabla\varphi_i^*\left(\nu_i^*\right)) \\
&= \sum_{i=1}^{N} (\nabla\phi_i^*(y_i) - \nabla\phi_i^*(y_i^\diamondsuit))^T \dot{y}_i(t) + \sum_{i=1}^{N} (\nabla\varphi_i^*(\nu_i) - \nabla\varphi_i^*(\nu_i^\diamondsuit))^T \dot{\nu}_i(t) \\
&= \sum_{i=1}^{N} (x_i - x_i^\diamondsuit)^T \dot{y}_i(t) + \sum_{i=1}^{N} (\sigma_i - \sigma_i^\diamondsuit))^T \dot{\nu}_i(t).
\end{aligned}$$

Here we employ the compact form defined in (s.10) to provide a more concise statement, leading to the derivation

$$\frac{d}{dt}V_1(t) = (\boldsymbol{x} - \boldsymbol{x}^\diamondsuit)^T(-\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) + \nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) + \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\right)^T \left(\Lambda\left(\boldsymbol{x}\right) - \nabla\Psi^*\left(\boldsymbol{\sigma}\right) + \nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}\right). \tag{s.21}$$

Meanwhile, by rearranging the terms in (s.21), we have

$$\begin{aligned}
\dot{V}_1 &= -(\boldsymbol{x} - \boldsymbol{x}^\diamondsuit)^T \boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\right)^T \left(-\Lambda\left(\boldsymbol{x}\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}\right)\right) + (\boldsymbol{x} - \boldsymbol{x}^\diamondsuit)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) + \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\right)^T \left(\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}\right) \\
&= -(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T F(\boldsymbol{z}) + (\boldsymbol{x} - \boldsymbol{x}^\diamondsuit)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit)^T \left(\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}\right),
\end{aligned} \tag{s.22}$$

where $\boldsymbol{z} = \operatorname{col}\{\boldsymbol{x}, \boldsymbol{\sigma}\}$ and $F(\boldsymbol{z}) = \operatorname{col}\{G(\boldsymbol{x}, \boldsymbol{\sigma}),\ -\Lambda\left(\boldsymbol{x}\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}\right)\}$ are defined in (s.6). Notice that (s.19) and (s.20) actually reveals that

$$(\boldsymbol{x} - \boldsymbol{x}^\diamondsuit)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) \leq 0, \quad (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit)^T \left(\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}\right) \leq 0. \tag{s.23}$$

Because $\boldsymbol{z}^\diamondsuit$ is a solution to VI$(\boldsymbol{\Xi}, F)$, we realize

$$\left(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit\right)^T F(\boldsymbol{z}^\diamondsuit) \geq 0. \tag{s.24}$$

Thus, (s.22) yields the further scaling that

$$\begin{aligned}
\dot{V}_1 &= -(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T F(\boldsymbol{z}) + (\boldsymbol{x} - \boldsymbol{x}^\diamondsuit)^T(\nabla\boldsymbol{\phi}(\boldsymbol{x}) - \boldsymbol{y}) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit)^T \left(\nabla\boldsymbol{\varphi}(\boldsymbol{\sigma}) - \boldsymbol{\nu}\right) \\
&\leq -(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T F(\boldsymbol{z}) \\
&= -(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T (F(\boldsymbol{z}) - F(\boldsymbol{z}^\diamondsuit)) - (\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T F(\boldsymbol{z}^\diamondsuit) \\
&\leq -(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T (F(\boldsymbol{z}) - F(\boldsymbol{z}^\diamondsuit)),
\end{aligned} \tag{s.25}$$

where the first inequality is due to (s.23) and the second inequality is due to (s.24). Now, we consider the term $(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T (F(\boldsymbol{z}) - F(\boldsymbol{z}^\diamondsuit))$ with details.

$$\begin{aligned}
&(\boldsymbol{z} - \boldsymbol{z}^\diamondsuit)^T (F(\boldsymbol{z}) - F(\boldsymbol{z}^\diamondsuit)) \\
&= (\boldsymbol{x} - \boldsymbol{x}^\diamondsuit)^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)) + \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\right)^T \left(-\Lambda\left(\boldsymbol{x}\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}\right) - (-\Lambda(\boldsymbol{x}^\diamondsuit) + \nabla\Psi^*(\boldsymbol{\sigma}^\diamondsuit))\right) \\
&= (\boldsymbol{x} - \boldsymbol{x}^\diamondsuit)^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^\diamondsuit, \boldsymbol{\sigma}^\diamondsuit)) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit\right)^T \left(\Lambda(\boldsymbol{x}) - \Lambda(\boldsymbol{x}^\diamondsuit)\right) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamondsuit)^T (\nabla\Psi^*(\boldsymbol{\sigma}) - \nabla\Psi^*(\boldsymbol{\sigma}^\diamondsuit)).
\end{aligned}$$

Due to the convexity of Ψ_i for $i \in \mathcal{I}$, the Legendre conjugate Ψ_i^* is also convex [82], which indicates

$$(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\diamondsuit})^T(\nabla\Psi^*(\boldsymbol{\sigma}) - \nabla\Psi^*(\boldsymbol{\sigma}^{\diamondsuit})) \geq 0.$$

Hence,

$$\begin{aligned} &(\boldsymbol{z} - \boldsymbol{z}^{\diamondsuit})^T(F(\boldsymbol{z}) - F(\boldsymbol{z}^{\diamondsuit})) \\ &\geq (\boldsymbol{x} - \boldsymbol{x}^{\diamondsuit})^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^{\diamondsuit}, \boldsymbol{\sigma}^{\diamondsuit})) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\diamondsuit}\right)^T (\Lambda(\boldsymbol{x}) - \Lambda(\boldsymbol{x}^{\diamondsuit})). \end{aligned} \tag{s.26}$$

Expanding the expression in (s.26),

$$\begin{aligned} &(\boldsymbol{x} - \boldsymbol{x}^{\diamondsuit})^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^{\diamondsuit}, \boldsymbol{\sigma}^{\diamondsuit})) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\diamondsuit}\right)^T (\Lambda(\boldsymbol{x}) - \Lambda(\boldsymbol{x}^{\diamondsuit})) \\ &= \sum\nolimits_{i=1}^{N}(x_i - x_i^{\diamondsuit})^T \left(\sum\nolimits_{k=1}^{q_i}[\sigma_i]_k \nabla_{x_i}\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) - \sum\nolimits_{k=1}^{q_i}[\sigma_i^{\diamondsuit}]_k \nabla_{x_i}\Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit})\right) \\ &\quad - \sum\nolimits_{i=1}^{N}\sum\nolimits_{k=1}^{q_i}([\sigma_i]_k - [\sigma_i^{\diamondsuit}]_k)\left(\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) - \Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit})\right). \end{aligned} \tag{s.27}$$

Rearranging (s.27), we have

$$\begin{aligned} &(\boldsymbol{x} - \boldsymbol{x}^{\diamondsuit})^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^{\diamondsuit}, \boldsymbol{\sigma}^{\diamondsuit})) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\diamondsuit}\right)^T (\Lambda(\boldsymbol{x}) - \Lambda(\boldsymbol{x}^{\diamondsuit})) \\ &= \sum\nolimits_{i=1}^{N}\sum\nolimits_{k=1}^{q_i}\left([\sigma_i]_k(x_i - x_i^{\diamondsuit})^T\nabla_{x_i}\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) - [\sigma_i^{\diamondsuit}]_k(x_i - x_i^{\diamondsuit})^T\nabla_{x_i}\Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit})\right) \\ &\quad - \sum\nolimits_{i=1}^{N}\sum\nolimits_{k=1}^{q_i}([\sigma_i]_k - [\sigma_i^{\diamondsuit}]_k)\left(\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) - \Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit})\right). \end{aligned} \tag{s.28}$$

By merging terms in (s.28), we have

$$\begin{aligned} &(\boldsymbol{x} - \boldsymbol{x}^{\diamondsuit})^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^{\diamondsuit}, \boldsymbol{\sigma}^{\diamondsuit})) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\diamondsuit}\right)^T (\Lambda(\boldsymbol{x}) - \Lambda(\boldsymbol{x}^{\diamondsuit})) \\ &= \sum\nolimits_{i=1}^{N}\sum\nolimits_{k=1}^{q_i}\left((x_i - x_i^{\diamondsuit})^T([\sigma_i]_k\nabla_{x_i}\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i})) + [\sigma_i]_k\Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit}) - [\sigma_i]_k\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i})\right) \\ &\quad + \sum\nolimits_{i=1}^{N}\sum\nolimits_{k=1}^{q_i}\left((x_i^{\diamondsuit} - x_i)^T([\sigma_i^{\diamondsuit}]_k\nabla_{x_i}\Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit})) + [\sigma_i^{\diamondsuit}]_k\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) - [\sigma_i^{\diamondsuit}]_k\Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit})\right). \end{aligned} \tag{s.29}$$

Recalling the definition in (9) that for $i \in \mathcal{I}$,

$$\sigma_i, \sigma_i^{\diamondsuit} \in \mathscr{C}_i^{+} = \{\sigma_i \in \Theta_i^* : \sum\nolimits_{k=1}^{q_i}[\sigma_i]_k\nabla_{x_i}^2\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) \succeq \kappa_x\boldsymbol{I}_n\}.$$

Hence, (s.29) satisfies

$$\begin{aligned} &\sum\nolimits_{i=1}^{N}\sum\nolimits_{k=1}^{q_i}\left((x_i - x_i^{\diamondsuit})^T([\sigma_i]_k\nabla_{x_i}\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i})) + [\sigma_i]_k\Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit}) - [\sigma_i]_k\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i})\right) \\ &\quad + \sum\nolimits_{i=1}^{N}\sum\nolimits_{k=1}^{q_i}\left((x_i^{\diamondsuit} - x_i)^T([\sigma_i^{\diamondsuit}]_k\nabla_{x_i}\Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit})) + [\sigma_i^{\diamondsuit}]_k\Lambda_{i,k}(x_i, \boldsymbol{x}_{-i}) - [\sigma_i^{\diamondsuit}]_k\Lambda_{i,k}(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit})\right) \\ &\geq \kappa_x\|\boldsymbol{x} - \boldsymbol{x}^{\diamondsuit}\|^2. \end{aligned}$$

which further yields

$$(\boldsymbol{z} - \boldsymbol{z}^{\diamondsuit})^T(F(\boldsymbol{z}) - F(\boldsymbol{z}^{\diamondsuit})) \geq \kappa_x\|\boldsymbol{x} - \boldsymbol{x}^{\diamondsuit}\|^2.$$

In this view, we can accordingly get

$$\dot{V}_1 \leq -\kappa_x\|\boldsymbol{x} - \boldsymbol{x}^{\diamondsuit}\|^2 \leq 0. \tag{s.30}$$

Since V_1 is radially unbounded in $\boldsymbol{x}(t)$ and $\boldsymbol{\sigma}(t)$, this implies that the trajectories of $\boldsymbol{x}(t)$ and $\boldsymbol{\sigma}(t)$ are bounded along the conjugate-based ODE (13).

Secondly, we show that $\boldsymbol{y}(t)$ and $\boldsymbol{\nu}(t)$ are bounded. Take another Lyapunov candidate function as $V_2 = \frac{1}{2}\|\boldsymbol{y}\|^2$, which is radially unbounded in $\boldsymbol{y}$. Along the trajectories of (s.10), the derivative of V_2 satisfies

$$\dot{V}_2 \leq \boldsymbol{y}^T(-\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) + \nabla\Phi(\boldsymbol{x})) - \|\boldsymbol{y}\|^2.$$

Because $\boldsymbol{x}$, $\boldsymbol{\sigma}$ have been proved to be bounded, it is clear that

$$\dot{V}_2 \leq -\|\boldsymbol{y}\|^2 + p_1\|\boldsymbol{y}\| = -2V_2 + p_1\sqrt{2V_2},$$

where p_1 is a positive constant. Analogously, take a third Lyapunov candidate function as $V_3 = \frac{1}{2}\|\boldsymbol{\nu}\|^2$, which is radially unbounded in $\boldsymbol{\sigma}$. Along the trajectories of (s.10), the derivative of V_3 satisfies

$$\begin{aligned} \dot{V}_3 &\leq \boldsymbol{\nu}^T(\Lambda(\boldsymbol{x}) - \nabla\Psi^*(\boldsymbol{\sigma}) + \nabla\varphi(\boldsymbol{\sigma}) - \boldsymbol{\nu}) - \|\boldsymbol{\nu}\|^2 \\ &\leq -\|\boldsymbol{\nu}\|^2 + p_2\|\boldsymbol{\nu}\| = -2V_3 + p_2\sqrt{2V_3}, \end{aligned}$$

with a positive constant p_2. Hence, it can be easily verified that V_2 and V_3 are bounded, so are $\boldsymbol{y}(t)$ and $\boldsymbol{\nu}(t)$.

(ii) Now let us investigate the set

$$Q \triangleq \left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}, \boldsymbol{\nu}) : \frac{d}{dt} V_1 = 0 \right\},$$

and take set I_v as its largest invariant subset. It follows from the invariance principle [83, Theorem 2.41] that $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}, \boldsymbol{\nu}) \to I_v$ as $t \to \infty$, and I_v is a positive invariant set. Then it follows from the derivation in (s.30) that

$$I_v \subseteq \left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\sigma}, \boldsymbol{\nu}) : \boldsymbol{x} = \boldsymbol{x}^{\diamondsuit} \right\}.$$

This indicates that any trajectory along ODE (13) results in the convergence of variable $\boldsymbol{x}$, that is, $\boldsymbol{x}(t) \to \boldsymbol{x}^{\diamondsuit}$ as $t \to \infty$. Moreover, if $\sigma_i^{\diamondsuit} = \nabla \Psi_i \left(\xi_i \right) |_{\xi_i = \Lambda_i \left(x_i^{\diamondsuit}, \boldsymbol{x}_{-i}^{\diamondsuit} \right)}$ for $i \in \mathcal{I}$, then the convergent point $\boldsymbol{x}^{\diamondsuit}$ indeed represents a global NE. So far, we have accomplished the proof. □

Based on the proof of Theorem 2, we further show the convergence rate of ODE (13). Here, we also reproduce Theorem 3 below for convenience:

Theorem 3 If $\mathscr{E}_i^{+}$ is nonempty and $\Psi_i\left(\cdot\right)$ is $\frac{1}{\kappa_\sigma}$-smooth for $i \in \mathcal{I}$, then (13) converges at an exponential rate, i.e.,

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^{\diamondsuit}\| \leq \sqrt{\frac{\tau}{\mu}} \|\boldsymbol{z}(0)\| \exp(-\frac{\kappa}{2\tau}\, t),$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$, $\kappa = \min\{\kappa_\sigma, \kappa_x\}$, $\tau = \max\{L_x/2\mu_x, L_\sigma/2\mu_\sigma\}$.

Proof. Take the same Lyapunov function as in Theorem 2:

$$V_1 = \sum_{i=1}^{N} D_{\phi_i^*}(y_i, y_i^{\diamondsuit}) + D_{\varphi_i^*}(\nu_i, \nu_i^{\diamondsuit}).$$

Recalling the analysis in Theorem 2, we have

$$V_1 \geq \mu \left(\left\| \boldsymbol{x} - \boldsymbol{x}^{\diamondsuit} \right\|^2 + \left\| \boldsymbol{\sigma} - \boldsymbol{\sigma}^{\diamondsuit} \right\|^2 \right) = \mu \left\| \boldsymbol{z} - \boldsymbol{z}^{\diamondsuit} \right\|^2, \tag{s.31}$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$.Based on the standard duality relations, the μ_x-strong convexity of generating function ϕ_i on Ω_i implies that its conjugate gradient $\nabla \phi_i^*$ is continuously differentiable on $\mathbb{R}^n$ with $1/\mu_x$-Lipschitz continuous gradient [84]. Thus,

$$\begin{aligned} \sum_{i=1}^{N} D_{\phi_i^*}(y_i, y_i^{\diamondsuit}) &= \sum_{i=1}^{N} \phi_i^*(y_i) - \phi_i^*(y_i^{\diamondsuit}) - \nabla \phi_i^*(y_i^{\diamondsuit})^T (y_i - y_i^{\diamondsuit}) \\ &\leq \frac{1}{2\mu_x} \sum_{i=1}^{N} \|y_i - y_i^{\diamondsuit}\|^2. \end{aligned} \tag{s.32}$$

Moreover, utilizing the duality relation $\nabla \phi_i(x_i) = y_i$ and $\nabla \phi_i(x_i^{\diamondsuit}) = y_i^{\diamondsuit}$, (s.32) yields

$$\begin{aligned} \sum_{i=1}^{N} D_{\phi_i^*}(y_i, y_i^{\diamondsuit}) &\leq \frac{1}{2\mu_x} \sum_{i=1}^{N} \|y_i - y_i^{\diamondsuit}\|^2 \\ &= \frac{1}{2\mu_x} \sum_{i=1}^{N} \|\nabla \phi_i(x_i) - \nabla \phi_i(x_i^{\diamondsuit})\|^2 \\ &\leq \frac{L_x}{2\mu_x} \|\boldsymbol{x} - \boldsymbol{x}^{\diamondsuit}\|^2, \end{aligned} \tag{s.33}$$

where the last inequality is due to the L_x-Lipschitz continuity of generating function ϕ_i. Analogously, $\nabla \varphi_i^*$ is $1/\mu_\sigma$-Lipschitz due to the μ_σ-strongly convexity of generating function φ_i on $\mathscr{E}_i^{+}$. Given that $\nabla \varphi_i(\sigma_i) = \nu_i$ and $\nabla \varphi_i(\sigma_i^{\diamondsuit}) = \nu_i^{\diamondsuit}$, we can further observe

$$\begin{aligned} \sum_{i=1}^{N} D_{\varphi_i^*}(\nu_i, \nu_i^{\diamondsuit}) &= \sum_{i=1}^{N} \varphi_i^*(\nu_i) - \varphi_i^*(\nu_i^{\diamondsuit}) - \nabla \varphi_i^*(\nu_i^{\diamondsuit})^T (\nu_i - \nu_i^{\diamondsuit}) \\ &\leq \frac{1}{2\mu_\sigma} \sum_{i=1}^{N} \|\nu_i - \nu_i^{\diamondsuit}\|^2 \\ &= \frac{1}{2\mu_\sigma} \sum_{i=1}^{N} \|\nabla \varphi_i(\sigma_i) - \nabla \varphi_i(\sigma_i^{\diamondsuit})\|^2 \\ &\leq \frac{L_\sigma}{2\mu_\sigma} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\diamondsuit}\|^2, \end{aligned}$$

where the last inequality is due to the L_σ-Lipschitz continuity of generating function φ_i. Therefore,

$$V_1 \le \sum_{i=1}^{N} D_{\phi_i^*}(y_i, y_i^\diamond) + D_{\varphi_i^*}(\nu_i, \nu_i^\diamond) \le \tau\|\boldsymbol{z} - \boldsymbol{z}^\diamond\|^2,$$

where $\tau = \max\{L_x/2\mu_x, L_\sigma/2\mu_\sigma\}$. Moreover, following the proof of Theorem 2, the derivate of V_1 satisfies

$$\dot{V}_1 \le -(\boldsymbol{z} - \boldsymbol{z}^\diamond)^T(F(\boldsymbol{z}) - F(\boldsymbol{z}^\diamond)).$$

Due to the definition of set $\mathscr{E}_i^+$ in (9) and the κ_σ-strongly convexity of Ψ_i^* for $i \in \mathcal{I}$, we have

$$\begin{aligned}
&(\boldsymbol{z} - \boldsymbol{z}^\diamond)^T(F(\boldsymbol{z}) - F(\boldsymbol{z}^\diamond)) \\
&= (\boldsymbol{x} - \boldsymbol{x}^\diamond)^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^\diamond, \boldsymbol{\sigma}^\diamond)) + \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond\right)^T\left(-\Lambda\left(\boldsymbol{x}\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}\right) - \left(-\Lambda\left(\boldsymbol{x}^\diamond\right) + \nabla\Psi^*\left(\boldsymbol{\sigma}^\diamond\right)\right)\right) \\
&= (\boldsymbol{x} - \boldsymbol{x}^\diamond)^T(\boldsymbol{G}(\boldsymbol{x}, \boldsymbol{\sigma}) - \boldsymbol{G}(\boldsymbol{x}^\diamond, \boldsymbol{\sigma}^\diamond)) - \left(\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond\right)^T\left(\Lambda\left(\boldsymbol{x}\right) - \Lambda(\boldsymbol{x}^\diamond)\right) + (\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond)^T(\nabla\Psi^*(\boldsymbol{\sigma}) - \nabla\Psi^*(\boldsymbol{\sigma}^\diamond)) \\
&\ge \kappa_x\|\boldsymbol{x} - \boldsymbol{x}^\diamond\|^2 + \kappa_\sigma\|\boldsymbol{\sigma} - \boldsymbol{\sigma}^\diamond\|^2 \\
&\ge \kappa\|\boldsymbol{z} - \boldsymbol{z}^\diamond\|^2,
\end{aligned}$$

where $\kappa = \min\{\kappa_\sigma, \kappa_x\}$. Therefore,

$$\dot{V}_1 \le -\kappa\|\boldsymbol{z} - \boldsymbol{z}^\diamond\|^2. \tag{s.34}$$

It follows from (s.31) and (s.34) that

$$\dot{V}_1 \le -\kappa\|\boldsymbol{z} - \boldsymbol{z}^\diamond\|^2 \le -\frac{\kappa}{\tau}V_1,$$

which actually yields the exponential convergence rate. In other words,

$$\mu\|\boldsymbol{z}(t) - \boldsymbol{z}^\diamond\|^2 \le V_1(\boldsymbol{z}(t)) \le V_1(\boldsymbol{z}(0))\exp(-\frac{\kappa}{\tau}\,t) \le \tau\|\boldsymbol{z}(0)\|^2\exp(-\frac{\kappa}{\tau}\,t).$$

Thus, we can also obtain

$$\|\boldsymbol{z}(t) - \boldsymbol{z}^\diamond\| \le \sqrt{\frac{\tau}{\mu}}\|\boldsymbol{z}(0)\|\exp(-\frac{\kappa}{2\tau}\,t),$$

which implies this conclusion. □

S.6 Bregman divergence and some inequalities

After the analysis of ODE (13), this section delves into exploring discrete algorithm 1, which stems from ODE (13). Also, we present several auxiliary results necessary for subsequent discussions.

First of all, the Bregman divergence associated with a generating function $h : \Xi \to \mathbb{R}$ is defined as

$$D_h(z', z) = h(z') - h(z) - (z' - z)^T\,\nabla h(z),\ \forall z, z' \in \Xi.$$

In what follows, we give basic bounds on the Bregman divergence. Firstly, the basic ingredient for these bounds is a generalization of the (Euclidean) law of cosines, which is known in the literature as the "three-point identity" [85]:

Lemma S2. Let the continuously differentiable generating function h be ω-strongly convex on set Ξ. For z, z', z^+ in Ξ, there holds

$$D_h(z', z^+) + D_h(z^+, z) = D_h(z', z) + \langle z' - z^+, \nabla h(z) - \nabla h(z^+)\rangle. \tag{s.35}$$

Proof. It follows from the definition of the Bregman divergence that

$$\begin{aligned}
D_h(z', z^+) &= h(z') - h\left(z^+\right) - \left(z' - z^+\right)^T\nabla h(z^+), \\
D_h(z^+, z) &= h(z^+) - h(z) - \left(z^+ - z\right)^T\nabla h(z), \\
D_h(z', z) &= h(z') - h\left(z\right) - \left(z' - z\right)^T\nabla h(z).
\end{aligned}$$

This lemma holds thus true by adding the first two equalities and subtracting the last one. □

Secondly, with the identity above, we have the following upper bound on a Bregman divergence.

Lemma S3. Let the continuously differentiable generating function h be ω-strongly convex on set Ξ. For z, z' in Ξ, and $z^+ = \Pi_\Xi^h(g) = \operatorname{argmin}_{z\in\Xi}\left\{-z^Tg + h(z)\right\}$, the following holds

$$D_h(z', z^+) \le D_h(z', z) - D_h(z^+, z) + (g - \nabla h(z))^T(z^+ - z'). \tag{s.36}$$

Proof. Based on the three-point identity (s.35), we obtain

$$D_h(z', z^+) + D_h(z^+, z) = D_h(z', z) + (z^+ - z')^T(\nabla h(z^+) - \nabla h(z)).$$

Rearranging these terms yields the following equation

$$D_h(z', z^+) = D_h(z', z) - D_h(z^+, z) + (z^+ - z')^T(\nabla h(z^+) - \nabla h(z)). \tag{s.37}$$

Moreover, given that $z^+ = \Pi_{\Xi}^{h}(g) = \mathrm{argmin}_{z\in\Xi}\left\{-z^T g + h(z)\right\}$, we can deduce from the optimality of z^+ and the convexity of h that

$$(-g + \nabla h(z^+))^T z^+ \le (-g + \nabla h(z^+))^T z',$$

which implies

$$(z^+ - z')^T \nabla h(z^+) \le (z^+ - z')^T g. \tag{s.38}$$

Thus, (s.36) holds by plugging (s.38) into (s.37). □

Before the end of this section, we introduce two classic inequalities in the following.

Lemma S4 (Fenchel's inequality). Take f as a continuous function on set C. Then the Fenchel conjugate f^* in dual space C^* is $f^*(b) = \sup_{a\in C}\left\{a^T b - f(a)\right\}$, which results in the following inequality

$$a^T b \le f(a) + f^*(b).$$

Lemma S5 (Jessen's inequality). Take f as a convex function on a convex set U, then

$$f(\sum_{l=1}^{k} \gamma_l x_l) \le \sum_{l=1}^{k} \gamma_l f(x_l),$$

where $x_1, \cdots, x_k \in U$ and $\gamma_1, \cdots, \gamma_k > 0$ with $\gamma_1 + \cdots + \gamma_k = 1$.

S.7 Proof of Theorem 4

With the basis mentioned above, we show the convergence analysis of Algorithm 1 on a class of N-player generalized monotone games. Suppose that $\Pi_{\Theta_i}^{\Psi_i}(\cdot)$ is κ_σ-strongly monotone, i.e., there exists a constant $\kappa_\sigma > 0$ such that $(\sigma_i - \sigma_i')^T(\Pi_{\Theta_i}^{\Psi_i}(\sigma_i) - \Pi_{\Theta_i}^{\Psi_i}(\sigma_i')) \ge \kappa_\sigma \|\sigma_i - \sigma_i'\|^2, \forall \sigma_i, \sigma_i' \in \mathscr{E}_i^+$. Here, we reproduce Theorem 4 below for convenience:

Theorem 4 If $\mathscr{E}_i^+$ is nonempty and $\Pi_{\Theta_i}^{\Psi_i}(\cdot)$ is κ_σ-strongly monotone, then Algorithm 1 converges at a rate of $\mathcal{O}(1/k)$ with step size $\alpha_k = \frac{2}{\kappa(k+1)}$, i.e.,

$$\|\boldsymbol{x}^k - \boldsymbol{x}^\Diamond\|^2 + \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^\Diamond\|^2 \le \frac{1}{k+1}\frac{M_1}{\mu^2\kappa^2},$$

where $\mu = \min\{\frac{\mu_x}{2}, \frac{\mu_\sigma}{2}\}$, $\kappa = \min\{\kappa_\sigma, \kappa_x\}$, and M_1 is a positive constant.

Proof. Take the collection of the Bregman divergence as

$$\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) \triangleq \sum_{i=1}^{N} D_{\phi_i}(x_i^\Diamond, x_i^{k+1}) + D_{\varphi_i}(\sigma_i^\Diamond, \sigma_i^{k+1}), \tag{s.39}$$

where

$$D_{\phi_i}(x_i^\Diamond, x_i^{k+1}) = \phi_i(x_i^\Diamond) - \phi_i(x_i^{k+1}) - \nabla\phi_i(x_i^{k+1})^T(x_i^\Diamond - x_i^{k+1}),$$
$$D_{\varphi_i}(\sigma_i^\Diamond, \sigma_i^{k+1}) = \varphi_i(\sigma_i^\Diamond) - \varphi_i(\sigma_i^{k+1}) - \nabla\varphi_i(\sigma_i^{k+1})^T(\sigma_i^\Diamond - \sigma_i^{k+1}).$$

Because ϕ_i is μ_x-strongly convex and φ_i is μ_σ-strongly convex for $i \in \mathcal{I}$, we obtain that

$$\begin{aligned}\Delta(\boldsymbol{z}^\Diamond, \boldsymbol{z}^{k+1}) &\ge \frac{\mu_x}{2}\sum_{i=1}^{N}\left\|x_i^{k+1} - x_i^\Diamond\right\|^2 + \frac{\mu_\sigma}{2}\sum_{i=1}^{N}\left\|\sigma_i^{k+1} - \sigma_i^\Diamond\right\|^2 \\ &\ge \mu\|\boldsymbol{z}^{k+1} - \boldsymbol{z}^\Diamond\|^2,\end{aligned} \tag{s.40}$$

where $\mu = \min\{\mu_x/2, \mu_\sigma/2\}$. Then, consider the term $D_{\phi_i}(x_i^\Diamond, x_i^{k+1})$ in (s.39). By employing three-point identity in Lemma S2, we obtain

$$D_{\phi_i}(x_i^\Diamond, x_i^{k+1}) = D_{\phi_i}(x_i^\Diamond, x_i^k) - D_{\phi_i}(x_i^{k+1}, x_i^k) + (\nabla\phi_i(x_i^{k+1}) - \nabla\phi_i(x_i^k))^T(x_i^{k+1} - x_i^\Diamond). \tag{s.41}$$

Denote

$$g_i = \nabla\phi_i(x_i^k) - \alpha_k \sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k, \boldsymbol{x}_{-i}^k).$$

According to Algorithm 1,

$$x_i^{k+1} = \Pi_{\Omega_i}^{\phi_i}(g_i) = \mathrm{argmin}_{x\in\Omega_i}\left\{-x^T g_i + \phi_i(x)\right\},$$

which implies

$$\begin{aligned} 0 &\leq \left(\nabla\phi_i\left(x_i^{k+1}\right)-g_i\right)^T x_i^{\diamondsuit}-\left(\left(\nabla\phi_i\left(x_i^{k+1}\right)\right)-g_i\right)^T x_i^{k+1} \\ &=\left(\nabla\phi_i\left(x_i^{k+1}\right)-g_i\right)^T (x_i^{\diamondsuit}-x_i^{k+1}). \end{aligned}$$

In addition,

$$(\nabla\phi_i(x_i^{k+1})^T(x_i^{k+1}-x_i^{\diamondsuit})\leq(\nabla\phi_i(x_i^k)-\alpha_k\sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k,\boldsymbol{x}_{-i}^k))^T(x_i^{k+1}-x_i^{\diamondsuit}).$$

Then (s.41) becomes

$$D_{\phi_i}(x_i^{\diamondsuit},x_i^{k+1})\leq D_{\phi_i}(x_i^{\diamondsuit},x_i^k)-D_{\phi_i}(x_i^{k+1},x_i^k)-\alpha_k(\sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k,\boldsymbol{x}_{-i}^k))^T(x_i^{k+1}-x_i^{\diamondsuit}). \tag{s.42}$$

Similarly, as for the term $D_{\varphi_i}(\sigma_i^{\diamondsuit},\sigma_i^{k+1})$ in (s.39), we get

$$D_{\varphi_i}(\sigma_i^{\diamondsuit},\sigma_i^{k+1})\leq D_{\varphi_i}(\sigma_i^{\diamondsuit},\sigma_i^k)-D_{\varphi_i}(\sigma_i^{k+1},\sigma_i^k)-\alpha_k(-\Lambda_i\left(x_i^k,\boldsymbol{x}_{-i}^k\right)+\xi_i^k)^T(\sigma_i^{k+1}-\sigma_i^{\diamondsuit}), \tag{s.43}$$

where $\xi_i^k=\Pi_{\Theta_i}^{\Psi_i}(\sigma_i^k)$. To proceed, combining (s.42) and (s.43) gives

$$\begin{aligned} \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^{k+1}) &= \sum_{i=1}^N D_{\phi_i}(x_i^{\diamondsuit},x_i^{k+1})+D_{\varphi_i}(\sigma_i^{\diamondsuit},\sigma_i^{k+1}) \\ &\leq \sum_{i=1}^N D_{\phi_i}(x_i^{\diamondsuit},x_i^k)-D_{\phi_i}(x_i^{k+1},x_i^k)-\alpha_k(\sigma_i^{kT}\nabla_{x_i}\Lambda_i(x_i^k,\boldsymbol{x}_{-i}^k))^T(x_i^{k+1}-x_i^{\diamondsuit}) \\ &\quad+\sum_{i=1}^N D_{\varphi_i}(\sigma_i^{\diamondsuit},\sigma_i^k)-D_{\varphi_i}(\sigma_i^{k+1},\sigma_i^k)-\alpha_k(-\Lambda_i\left(x_i^k,\boldsymbol{x}_{-i}^k\right)+\xi_i^k)^T(\sigma_i^{k+1}-\sigma_i^{\diamondsuit}), \\ &= \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k+1}-\boldsymbol{z}^{\diamondsuit})-\Delta(\boldsymbol{z}^{k+1},\boldsymbol{z}^k). \end{aligned}$$

Hence,

$$\begin{aligned} \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^{k+1}) &\leq \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k+1}-\boldsymbol{z}^{\diamondsuit})-\Delta(\boldsymbol{z}^{k+1},\boldsymbol{z}^k) \\ &= \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})+\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{k+1})-\Delta(\boldsymbol{z}^{k+1},\boldsymbol{z}^k) \\ &\leq \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})+\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{k+1})-\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2, \end{aligned}$$

where the last inequality is due to a similar property in (s.40). On this basis, by additionally employing Fenchel's inequality and subsituting f in Lemma S4 with $\frac{1}{2}\|\cdot\|$, we obtain

$$\begin{aligned} \alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{k+1}) &\leq \frac{(2\mu)}{2}\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2+\frac{(2\mu)^{-1}}{2}\alpha_k^2\|F(\boldsymbol{z}^k)^T\|_*^2 \\ &= \mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2+\frac{1}{4\mu}\alpha_k^2\|F(\boldsymbol{z}^k)^T\|_*^2 \\ &= \mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2+\frac{1}{4\mu}\alpha_k^2\|F(\boldsymbol{z}^k)^T\|^2, \end{aligned}$$

where the last equality arises from the conjugate norm of the ℓ_2 norm being the ℓ_2 norm itself.

Hence, we can make further scaling so that

$$\begin{aligned} \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^{k+1}) &\leq \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})+\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{k+1})-\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2 \\ &\leq \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})+\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)^T\|^2-\mu\|\boldsymbol{z}^k-\boldsymbol{z}^{k+1}\|^2 \\ &= \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_kF(\boldsymbol{z}^k)^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2 \\ &= \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_k(F(\boldsymbol{z}^k)-F(\boldsymbol{z}^{\diamondsuit}))^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})-\alpha_kF(\boldsymbol{z}^{\diamondsuit})^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2 \\ &\leq \Delta(\boldsymbol{z}^{\diamondsuit},\boldsymbol{z}^k)-\alpha_k(F(\boldsymbol{z}^k)-F(\boldsymbol{z}^{\diamondsuit}))^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})+\frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2, \end{aligned} \tag{s.44}$$

where the last inequality is true because $\boldsymbol{z}^{\diamondsuit}$ is a solution to VI($\boldsymbol{\Xi}$, F). Moreover, with κ_σ-strongly monotonicity of operator $\Pi_{\Theta_i}^{\Psi_i}(\cdot)$, there holds the inequality

$$(F(\boldsymbol{z}^k)-F(\boldsymbol{z}^{\diamondsuit}))^T(\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit})\geq\kappa\|\boldsymbol{z}^k-\boldsymbol{z}^{\diamondsuit}\|^2,$$

where $\kappa = \min\{\kappa_x, \kappa_\sigma\}$. Then

$$\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \leq \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k \kappa \|\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit\|^2 + \frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2.$$

Denote $\eta_k = \kappa\alpha_k$ with $\eta_0 = 1$. We can verify that

$$\frac{1-\eta_{k+1}}{\eta_{k+1}^2} \leq \frac{1}{\eta_k^2}, \quad \forall k \geq 0.$$

Then, with the substitute above,

$$\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \leq \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \eta_k \|\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit\|^2 + \frac{\eta_k^2}{4\kappa^2\mu}\|F(\boldsymbol{z}^k)\|^2. \tag{s.45}$$

On the one hand, recalling the property of the Bregman divergence [62], [86], we have

$$\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}) \leq \frac{1}{2}\|\boldsymbol{z} - \boldsymbol{z}^\diamondsuit\|^2 \leq \|\boldsymbol{z} - \boldsymbol{z}^\diamondsuit\|^2, \quad \forall \boldsymbol{z}, \boldsymbol{z}^\diamondsuit \in \boldsymbol{\Xi}.$$

On the other hand, since the stationary points are with finite values, the set $\mathscr{E}_i^+$ for $i \in \mathcal{I}$ can be regarded as bounded without loss of generality. Combined with the compactness of the feasible set Ω_i for $i \in \mathcal{I}$, there exists a finite constant $M_1 > 0$ such that $\|F(\boldsymbol{z})\|^2 \leq M_1$. Based on this foundation, we derive

$$\begin{aligned}\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) &\leq \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \eta_k \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) + \frac{\eta_k^2}{4\kappa^2\mu}\|F(\boldsymbol{z}^k)\|^2 \\ &\leq (1-\eta_k)\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) + \frac{\eta_k^2}{4\kappa^2\mu}\|F(\boldsymbol{z}^k)\|^2 \\ &\leq (1-\eta_k)\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) + \frac{\eta_k^2}{4\kappa^2\mu}M_1.\end{aligned} \tag{s.46}$$

Multiplying both sides of the relation above by $1/\eta_k^2$, and recalling the property $\frac{1-\eta_{k+1}}{\eta_{k+1}^2} \leq \frac{1}{\eta_k^2}$, we obtain

$$\begin{aligned}\frac{1}{\eta_k^2}\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) &\leq \frac{1-\eta_k}{\eta_k^2}\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) + \frac{M_1}{4\kappa^2\mu} \\ &\leq \frac{1}{\eta_{k-1}^2}\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) + \frac{M_1}{4\kappa^2\mu}.\end{aligned}$$

Hence, summing up these inequalities over $k, \cdots, 1$ with $\eta_0 = 1$, that is,

$$\frac{1}{\eta_k^2}\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \leq \Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^1) + k\frac{M_1}{4\kappa^2\mu}.$$

Additionally, by taking $k = 1$ and $\eta_0 = 1$ in (s.46), $\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^1) \leq \frac{\eta_k^2 M_1}{4\kappa^2\mu}$. Therefore, recalling the step size setting $\eta_k = \kappa\alpha_k = 2/(k+1)$, for all $k \geq 1$, we get

$$\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \leq \eta_k^2(k+1)\frac{M_1}{4\kappa^2\mu} = \frac{1}{k+1}\frac{M_1}{\mu\kappa^2}. \tag{s.47}$$

Recall

$$\Delta(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) \geq \mu\|\boldsymbol{z}^{k+1} - \boldsymbol{z}^\diamondsuit\|^2 = \mu(\|\boldsymbol{x}^k - \boldsymbol{x}^\diamondsuit\|^2 + \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^\diamondsuit\|^2).$$

Therefore, we are finally rewarded with

$$\|\boldsymbol{x}^k - \boldsymbol{x}^\diamondsuit\|^2 + \|\boldsymbol{\sigma}^k - \boldsymbol{\sigma}^\diamondsuit\|^2 \leq \frac{1}{k+1}\frac{M_1}{\mu^2\kappa^2},$$

which completes the proof. □

S.8 Proof of Theorem 5

As §5, the nonconvex N-player potential game in (15) is endowed with a unified complementary function in (16), that is,

$$\Gamma(x_i, \sigma, \boldsymbol{x}_{-i}) = \sigma^T \Lambda(x_i, \boldsymbol{x}_{-i}) - \Psi^*(\sigma).$$

Thus, we can employ the gradient information of this unified complementary function in algorithm iterations, to reduce the computational cost in Algorithm 1. Accordingly, we can rewrite Algorithm 1 for potential games in the following Algorithm 2.

Algorithm 2

Input: Step size $\{\alpha_k\}$, proper generating functions ϕ_i on Ω_i and φ on $\mathscr{E}^+$.
Initialize: Set $\sigma^0 \in \mathscr{E}^+, x_i^0 \in \Omega_i,\ i \in \{1,\ldots,N\}$,

for $k = 1, 2, \cdots$ **do**
  compute the unified conjugate of Ψ: $\xi^k = \Pi_\Theta^\Psi(\sigma^k)$
  update the unified canonical dual variable:
    $\sigma^{k+1} = \Pi^\varphi_{\mathscr{E}^+}(\nabla\varphi(\sigma^k) + \alpha_k(\Lambda\left(x_i^k, \boldsymbol{x}_{-i}^k\right) - \xi^k))$
  for $i = 1, \cdots N$ **do**
    update the decision variable of player i:
    $x_i^{k+1} = \Pi^{\phi_i}_{\Omega_i}(\nabla\phi_i(x_i^k) - \alpha_k \sigma^{kT}\nabla_{x_i}\Lambda\left(x_i^k, \boldsymbol{x}_{-i}^k\right))$
  end for
end for

Similarly, define $\boldsymbol{z} = \operatorname{col}\{\boldsymbol{x}, \sigma\}$, and the simplified pseudo-gradient of (16) as

$$F(\boldsymbol{z}) \triangleq \begin{bmatrix} \operatorname{col}\left\{\sigma^T \nabla_{x_i}\Lambda\left(x_i, \boldsymbol{x}_{-i}\right)\right\}_{i=1}^N \\ -\Lambda\left(x_i, \boldsymbol{x}_{-i}\right) + \nabla\Psi^*\left(\sigma\right) \end{bmatrix} \triangleq \begin{bmatrix} G(\boldsymbol{x}, \sigma) \\ -\Lambda\left(\boldsymbol{x}\right) + \nabla\Psi^*\left(\sigma\right) \end{bmatrix}.$$

Consider the weighted averaged iterates in the course of k iterates as

$$\hat{\boldsymbol{x}}^k = \frac{\sum_{j=1}^k \alpha_j \boldsymbol{x}^j}{\sum_{j=1}^k \alpha^j}, \quad \hat{\sigma}^k = \frac{\sum_{j=1}^k \alpha_j \sigma^j}{\sum_{j=1}^k \alpha^j}.$$

Then we show the convergence rate of Algorithm 2 (or Algorithm 1 in potential games). We rewrite Theorem 5 below for convenience:

Theorem 5 If $\mathscr{E}^+$ is nonempty and players' payoffs are subject to the potential function in (15), then Algorithm 1 converges at a rate of $\mathcal{O}(1/\sqrt{k})$ with step size $\alpha_k = \frac{2\mu d}{M_2\sqrt{k}}$, i.e.,

$$\Gamma(\hat{\boldsymbol{x}}^k, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \hat{\sigma}^k) \leq \frac{1}{\sqrt{k}}\sqrt{\frac{d}{\mu}}M_2,$$

where $\mu = \min\{\frac{\mu_x}{2}, \frac{\mu_\sigma}{2}\}$, and d, M_2 are two positive constants.

Proof Take another collection of the Bregman divergence as

$$\widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}) \triangleq D_\varphi(\sigma^\diamondsuit, \sigma) + \sum_{i=1}^N D_{\phi_i}(x_i^\diamondsuit, x_i). \tag{s.48}$$

Working as in the proof of Theorem 4, we derive the following inequality by three-point identity and Fenchel's inequality:

$$\begin{aligned} \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) &\leq \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^{k+1} - \boldsymbol{z}^\diamondsuit) - \widetilde{\Delta}(\boldsymbol{z}^{k+1}, \boldsymbol{z}^k) \\ &\leq \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit) + \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^{k+1}) - \widetilde{\Delta}(\boldsymbol{z}^{k+1}, \boldsymbol{z}^k) \\ &\leq \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit) + \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^{k+1}) - \mu\|\boldsymbol{z}^k - \boldsymbol{z}^{k+1}\|^2 \\ &\leq \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit) + \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^{k+1}) - \mu\|\boldsymbol{z}^k - \boldsymbol{z}^{k+1}\|^2 \\ &\leq \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit) + \frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2. \end{aligned} \tag{s.49}$$

Moreover, according to $\sigma \in \mathscr{E}^+$ in (9),

$$(\boldsymbol{x}^\diamondsuit - \boldsymbol{x}^k)^T G(\boldsymbol{x}^k, \sigma^k) \leq \sigma^{kT}(\Lambda\left(\boldsymbol{x}^\diamondsuit\right) - \Lambda\left(\boldsymbol{x}^k\right)).$$

As a result,

$$\begin{aligned} \left\langle F(\boldsymbol{z}^k), \boldsymbol{z}^\diamondsuit - \boldsymbol{z}^k \right\rangle &= (\boldsymbol{x}^\diamondsuit - \boldsymbol{x}^k)^T G(\boldsymbol{x}^k, \sigma^k) + (\sigma^\diamondsuit - \sigma^k)^T(-\Lambda\left(\boldsymbol{x}^k\right) + \nabla\Psi^*\left(\sigma^k\right)) \\ &\leq \sigma^{kT}\Lambda\left(\boldsymbol{x}^\diamondsuit\right) - \Psi^*\left(\sigma^k\right) - (\sigma^{\diamondsuit T}\Lambda\left(\boldsymbol{x}^{kT}\right) - \Psi^*\left(\sigma^\diamondsuit\right)) \\ &= \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^k) - \Gamma(\boldsymbol{x}^k, \sigma^\diamondsuit). \end{aligned} \tag{s.50}$$

By substituting (s.50) into (s.49) and rearranging the terms therein, we have

$$\begin{aligned} \alpha_k(\Gamma(\boldsymbol{x}^k, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^k)) &\leq \alpha_k F(\boldsymbol{z}^k)^T(\boldsymbol{z}^k - \boldsymbol{z}^\diamondsuit) \\ &\leq \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^k) - \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^{k+1}) + \frac{\alpha_k^2}{4\mu}\|F(\boldsymbol{z}^k)\|^2. \end{aligned}$$

Meanwhile, as the stationary points have finite values, the set $\mathscr{E}^+$ can be considered bounded without loss of generality. Together with the compactness of Ω_i for $i \in \mathcal{I}$, there exists finite constants $d > 0$ and $M_2 > 0$ such that $\widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^1) \leq d$ and $\|F(\boldsymbol{z})\|^2 \leq M_2$. Then, summing up the above inequalities over $1, \cdots, k$ yields

$$\sum_{j=1}^{k} \alpha_j \left(\Gamma(\boldsymbol{x}^j, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^j)\right) \leq \widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^1) + \frac{\sum_{j=1}^{k} \alpha_j^2 M_2^2}{4\mu}. \tag{s.51}$$

For more intuitive presentation, we denote the weight by $\lambda_j = \frac{\alpha_j}{\sum_{l=1}^{k} \alpha_l}$. Then (s.51) yields

$$\begin{aligned}
\frac{\widetilde{\Delta}(\boldsymbol{z}^\diamondsuit, \boldsymbol{z}^1) + (4\mu)^{-1} M_2^2 \sum_{j=1}^{k} \alpha_j^2}{\sum_{j=1}^{k} \alpha_j} &\geq \sum_{j=1}^{k} \frac{\alpha_j}{\sum_{l=1}^{k} \alpha_l} \left(\Gamma(\boldsymbol{x}^j, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^j)\right) \\
&= \sum_{j=1}^{k} \lambda_j \left(\Gamma(\boldsymbol{x}^j, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sigma^j)\right) \\
&\geq \Gamma(\sum_{j=1}^{k} \lambda_j \boldsymbol{x}^j, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \sum_{j=1}^{k} \lambda_j \boldsymbol{\sigma}^j) \\
&= \Gamma(\hat{\boldsymbol{x}}^k, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \hat{\sigma}^k),
\end{aligned}$$

where the last inequality is true due to Jensen's inequality. Since the step size satisfies $\alpha_k = 2\sqrt{\mu d}/M_2\sqrt{k}$, we finally derive that

$$\Gamma(\hat{\boldsymbol{x}}^k, \sigma^\diamondsuit) - \Gamma(\boldsymbol{x}^\diamondsuit, \hat{\sigma}^k) \leq \frac{1}{\sqrt{k}} \sqrt{\frac{d}{\mu}} M_2,$$

which indicates the conclusion. □

S.9 The computation of $\mathscr{E}_i^+$

It follows from Theorem 1 that verifying the existence of a global Nash equilibrium (NE) entails computing the feasible set $\mathscr{E}_i^+$ of σ_i. Nonetheless, the derivation and computation of $\mathscr{E}_i^+$ are not complicated in practical problems. Here we consider two examples. One is the sensor network localization problem [15], [51], where the structure of $\mathscr{E}_i^+$ can be obtained as unit square constraints. The other is the storage control problem [87], [88], where $\mathscr{E}_i^+$ is defined by linear constraints.

In the sensor network localization problem [15], [51], non-anchor node i selects its localization strategy x_i from $\Omega_i \subseteq \mathbb{R}^n$, which is represented as a unit square form. The payoff of non-anchor node i is

$$J_i(x_i, \boldsymbol{x}_{-i}) = \Psi_i(\Lambda_i(x_i, \boldsymbol{x}_{-i})) = \sum_{j \in \mathcal{N}_s^i} (\|x_i - x_j\|^2 - d_{ij}^2)^2 + \frac{\kappa}{2}\|x_i\|^2,$$

where $\Psi_i = \sum_{j \in \mathcal{N}_s^i} \Lambda_{i,j}^T \Lambda_{i,j}$ and $\Lambda_{i,j} = \|x_i - x_j\|^2 - d_{ij}$. Denote $\sigma_i = \mathrm{col}\{\sigma_{i,l}\}_{l=1}^{|\mathcal{N}_s^i|} \in \Theta_i^* \subseteq \mathbb{R}^{q_i}$, where $q_i = |\mathcal{N}_s^i|$. The complementary function is

$$\Gamma_i(x_i, \boldsymbol{x}_{-i}, \sigma_i) = \sum\nolimits_{l=1}^{|\mathcal{N}_s^i|} (\sigma_{i,l}(\|x_i - x_l\|^2 - d_{il}^2)^2 - \frac{\sigma_{i,l}^2}{4}) + \frac{\kappa}{2}\|x_i\|^2.$$

Then, we have $\mathscr{E}_i^+ = \Theta_i^* \cap \{\sigma_i : P_i(\sigma_i) + \kappa_x \boldsymbol{I}_n \succeq \kappa_x \boldsymbol{I}_n\}$. This implies $P_i(\sigma_i) \succeq \boldsymbol{0}_n$. Moreover, since the global NE $\boldsymbol{x}^\diamondsuit$ represents the localization accuracy for all sensors satisfying $\|x_i^\diamondsuit - x_j^\diamondsuit\|^2 - d_{ij}^2 = 0$ for $i, j \in \mathcal{N}_s$ [51], we have $\sigma_{il}^\diamondsuit = \nabla_{\xi_{il}} \Psi_i(\xi_i) \mid_{\xi_i = \Lambda_i(x_i^\diamondsuit, \boldsymbol{x}_{-i}^\diamondsuit)} = 2(\|x_i^\diamondsuit - x_l^\diamondsuit\|^2 - d_{il}^2) = 0$. On this basis, since $\boldsymbol{0}_{q_i} \in \mathscr{E}_i^+$, $\mathscr{E}_i^+$ can be replaced by some unit square constraints $[0, M]^{q_i}$, where M is a constant related to the upper bound of Ω_i.

Consider another example in storage control [87], [88]. Storage i selects its charge action $x_i \in \Omega_i \subseteq \mathbb{R}^n$ to minimize its electricity cost J_i within a given period n time slots, i.e.,

$$J_i(x_i, \boldsymbol{x}_{-i}) = \sum_{t=1}^{n} \left(\left(D + \sum\nolimits_{i=1}^{N} x_{i,t} + \lambda x_{i,t}^2 \right) + h \right) \cdot \left(x_{i,t} + \lambda x_{i,t}^2 \right),$$

where $D \in \mathbb{R}$ represents the background demand at time t, $x_{i,t} + \lambda x_{i,t}^2$ represents the energy purchased by storage i at time t, $\lambda \in \mathbb{R}$ reflects the degree of quadratic charging loss during charging, and h is the linear coefficient. The admissible control feasible Ω_i is subject to linear constraints $\Omega_i = \{x_i | A_i x_i \leq E_i\}$, as seen in [87]. We reformulate this problem with a potential game model, where the potential function is

$$\begin{aligned} H(x_i, \boldsymbol{x}_{-i}) =& \Psi(\Lambda(x_i, \boldsymbol{x}_{-i})) \\ =& \sum_{t=1}^{n}\sum_{i=1}^{N}(D+h)\left(x_{i,t}+\lambda x_{i,t}^2\right) + \frac{1}{2}\sum_{t=1}^{n}\sum_{i=1}^{N}\left(x_{i,t}+\lambda x_{i,t}^2\right)^2 + \frac{1}{2}\sum_{t=1}^{n}(\sum_{i=1}^{N} x_{i,t}+\lambda x_{i,t}^2)^2, \end{aligned}$$

where $\Psi = \frac{1}{2}\Lambda^T\Lambda + \sum_{t=1}^{n}\sum_{i=1}^{N}(D+h)\left(x_{i,t}+\lambda x_{i,t}^2\right)$ and $\Lambda = \operatorname{col}\left\{\operatorname{col}\{x_{1,t}+\lambda x_{1,t}^2\}_{t=1}^n, \cdots, \operatorname{col}\{x_{N,t}+\lambda x_{N,t}^2\}_{t=1}^n, \operatorname{col}\{\sum_{i=1}^{N} x_{i,t}+\lambda x_{i,t}^2\}_{t=1}^n\right\} \subseteq \mathbb{R}^{n(N+1)}$. Denote $\sigma \in \Theta^* \subseteq \mathbb{R}^{n(N+1)}$. Based on the canonical transformation, we have

$$\Gamma(\boldsymbol{x},\sigma) = \sigma^T\Lambda(\boldsymbol{x}) - \frac{1}{2}\sigma^2 + \sum_{t=1}^{n}\sum_{i=1}^{N}(D+h)\left(x_{i,t}+\lambda x_{i,t}^2\right).$$

Then, $\mathscr{E}^+ = \Theta^* \cap \{\sigma : P(\sigma) + 2\lambda(D+h)\boldsymbol{I}_{nN} \succeq \kappa_x \boldsymbol{I}_{nN}\}$, where Θ^* is a compact set with linear constraints and $P(\sigma)$ is a diagonal matrix, i.e.,

$$P(\sigma) = \begin{bmatrix} 2\lambda(\sigma_1+\sigma_{nN+1}) & 0 & \cdots & 0 \\ 0 & 2\lambda(\sigma_2+\sigma_{nN+2}) & \cdots & 0 \\ 0 & 0 & \cdots & 2\lambda(\sigma_{nN}+\sigma_{n(N+1)}) \end{bmatrix}.$$

This implies that $\mathscr{E}^+$ can be transformed into linear constraints.

S.10 Derivation of robust neural network formulation

The log-sum-exp function introduced in Exam. 2 is

$$\log[1+\exp(-a_1 x_i^T A x_i - 2a_1\sum\nolimits_{j\neq i,j=1}^{N} x_i^T A x_j - a_1\sum\nolimits_{j\neq i,j=1}^{N} x_j^T A x_j - a_2\beta_1^T x_i - a_2\sum\nolimits_{j\neq i,j=1}^{N}\beta_1^T x_j)]. \quad \text{(s.52)}$$

Consider a binary classification task in RNN. We begin with the conventional representation of log-sum-exp, and show the derivation of the new representation (s.52) from it. Take $N=2$ for an illustration. The cross-entropy loss function of the binary classification task is

$$Loss = -(\beta_2\log(s) + (1-\beta_2)\log(1-s))$$

where $s\in\mathbb{R}$ is the activation function of the output $z=(x_1+x_2)^T\beta_1$. Here, $\beta_1\in\mathbb{R}^n$ is an adversarial example of a clean one $\beta_0\in\mathbb{R}^n$, $\beta_2\in\mathbb{R}$ is the label of the clean $\beta_0\in\mathbb{R}^n$, $x_1\in\Omega_1\subseteq\mathbb{R}^n$ is the neural network parameter and x_2 is the adversarial weight perturbation [52], [53].

Take $s=\sigma(z)=1/(1+\exp(-a_1z^2-a_2z))$ as a quadratic sigmoid activation function [70] with two parameters $a_1>0$ and $a_2>0$. Then

$$\begin{aligned} Loss =& -\beta_2\log(s) - \log(1-s) + \beta_2\log(1-s) \\ =& -\beta_2\log(\frac{1}{1+\exp(-a_1z^2-a_2z)}) - \log(\frac{\exp(-a_1z^2-a_2z)}{1+\exp(-a_1z^2-a_2z)}) + \beta_2\log(\frac{\exp(-a_1z^2-a_2z)}{1+\exp(-a_1z^2-a_2z)}) \\ =& \beta_2\log(1+\exp(-a_1z^2-a_2z)) + a_1z^2 + a_2z + \log(1+\exp(-a_1z^2-a_2z)) - \beta_2(a_1z^2+a_2z) \\ & -\beta_2\log(1+\exp(-a_1z^2-a_2z)) \\ =& (1-\beta_2)(a_1z^2+a_2z) + \log(1+\exp(-a_1z^2-a_2z)) \\ =& (1-\beta_2)(a_1x_1^TA_lx_1 + 2a_1x_1^TA_lx_2 + a_1x_2^TA_lx_2 + a_2\beta_1^Tx_1 + a_2\beta_1^Tx_2) \\ & + \log[1+\exp(-a_1x_1^TAx_1 - 2a_1x_1^TAx_2 - a_1x_2^TAx_2 - a_2\beta_1^Tx_1 - a_2\beta_1^Tx_2)], \end{aligned}$$

which yields (s.52).