
Tighter Generalization Bounds for Iterative Privacy-Preserving Algorithms

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Abstract

This paper studies the relationship between generalization and privacy preservation of machine learning in two steps. We first establish an alignment between the two facets for any learning algorithm. We prove that (ϵ, δ) -differential privacy implies an on-average generalization bound for a multi-sample-set learning algorithm, which further leads to a high-probability bound for any learning algorithm. We then investigate how the iterative nature shared by most learning algorithms influences privacy preservation and further generalization. Three composition theorems are proved to approximate the differential privacy of an iterative algorithm through the differential privacy of its every iteration. Integrating the above two steps, we eventually deliver generalization bounds for iterative learning algorithms. Our results are strictly tighter than the existing works. Particularly, our generalization bounds do not rely on the model size which is prohibitively large in deep learning. Experiments of MLP, VGG, and ResNet on MNIST, CIFAR-10, and CIFAR-100 are in full agreement with our theory. The theory applies to a wide spectrum of learning algorithms. In this paper, it is applied to the Gaussian mechanism as an example.

1 INTRODUCTION

Generalization to unseen data and privacy preservation are two increasingly important facets of machine learning. Specifically, good generalization guarantees that an algorithm learns the underlying patterns in the training data rather than just memorizes the data [Vapnik, 2013, Mohri et al., 2018]. In this way, good generalization abilities provide confidence that the models trained on existing data

can be applied to similar but unseen scenarios. Additionally, massive personal data has been collected, such as financial and medical records. How to discover the highly valuable population knowledge carried in the data while protecting the highly sensitive individual privacy has profound importance [Dwork and Roth, 2014, Pittaluga and Koppal, 2016].

This paper investigates the relationship between generalization and privacy preservation in machine learning algorithms by the following two steps: (1) exploring the relationship between generalization and privacy preservation in any learning algorithm; and (2) analyzing how the iterative nature shared by most learning algorithms would influence the privacy-preserving ability and further the generalizability.

We first prove two theorems that upper bound the generalization error of an learning algorithm via its differential privacy. Specifically, we prove a high-probability upper bound for the generalization error,

$$\text{Gen}_{S, \mathcal{A}(S)} = \mathcal{R}_{\mathcal{D}}(\mathcal{A}(S)) - \hat{\mathcal{R}}_S(\mathcal{A}(S)),$$

where S is the training set sampled i.i.d. from some distribution \mathcal{D} , $\mathcal{A}(S)$ is the hypothesis learned by algorithm \mathcal{A} on S , $\mathcal{R}_{\mathcal{D}}(\mathcal{A}(S))$ is the expected risk, and $\hat{\mathcal{R}}_S(\mathcal{A}(S))$ is the empirical risk. This bound is established based on a novel on-average generalization bound for any (ϵ, δ) -differentially private multi-sample-set learning algorithm. These results indicate that the algorithms with a good privacy-preserving ability also have a good generalizability. We, therefore, can expect to design novel learning algorithms for better generalizability by enhancing its privacy-preserving ability.

We then studied how the iterative nature shared by most learning algorithms influences the privacy-preserving ability. Generally, the privacy-preserving ability of an iterative algorithm degenerates along iterations, since the amount of leaked information cumulates when the algorithm is progressing. To capture this degenerative property, we further prove three composition theorems that calculate the differential privacy of any iterative algorithm via the differential privacy of its every iteration. Combining with the

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established relationship between generalization and privacy preservation, our composition theorems help characterizing the generalizabilities of iterative learning algorithms.

Our results considerably extend the current understanding of the relationship between generalization and privacy preservation in iterative learning algorithms.

Existing works [Dwork et al., 2015, Nissim and Stemmer, 2015, Oneto et al., 2017] have proved some high-probability generalization bounds in the following form,

$$\mathbb{P}(|\text{Gen}_{S, \mathcal{A}(S)}| > a) < b, \quad (1)$$

where a and b are two positive constant real numbers. Our high-probability bound is strictly tighter than the current tightest results by [Nissim and Stemmer, 2015] which only holds for $\varepsilon \leq \frac{1}{10}$ from two aspects: (1) our bound tightens the term a from 13ε to 4ε and the term b from $\frac{2\delta}{\varepsilon} \log\left(\frac{2}{\varepsilon}\right)$ to $\frac{2e^{-1.7\varepsilon}\delta}{\varepsilon} \log\left(\frac{2}{\varepsilon}\right)$, and (2) our bound further cover the case when $\varepsilon > \frac{1}{10}$. These improvements are significant in practice because the factor ε can be as large as 10 in the experiments by [Abadi et al., 2016]. Also, the bounds by [Nissim and Stemmer, 2015] are only for binary classification, while ours apply to any differentially private learning algorithm.

There are also existing literature on the differential privacy composition bound [Dwork and Roth, 2014, Kairouz et al., 2017], and the current state-of-art bound is given by [Kairouz et al., 2015] using "privacy region" technique. The approximation of factor δ in our composition theorems is tighter than that in [Kairouz et al., 2017] by

$$\delta \frac{e^\varepsilon - 1}{e^\varepsilon + 1} \left(T - \left\lceil \frac{\varepsilon'}{\varepsilon} \right\rceil \right),$$

where T is number of iterations, while the estimate of ε' remains the same. This improvement is significant because the iteration number T can be considerably large in practice. This helps our composition theorems to further tighten our generalization bounds for iterative learning algorithms considerably.

We trained MLPs on the MNIST dataset [LeCun, 1998], and VGG-16 [Simonyan and Zisserman, 2014] and ResNet-18 [He et al., 2016] on the CIFAR-10 and CIFAR-100 datasets [Krizhevsky et al., 2009]. Membership inference attack [Yeom et al., 2018, Shokri et al., 2017] is performed in every epoch. The membership inference attack accuracy and the generalization error (the difference between test error and training error) in every epoch are collected for evaluating the privacy-preserving ability and the generalizability, respectively. The collected (membership inference attack accuracy, generalization error) pairs, (membership inference attack accuracy, training time) pairs, and (generalization error, training time) pairs are divided into groups according to the neural architecture and the dataset. Spearman's rank

order correlation test [Spearman, 1987] is performed to all groups. The correlation coefficients and p -values show three statistically significant correlations: (1) the positive correlation between the generalization and privacy preservation; (2) the negative correlation between the generalization and training time; and (3) the negative correlation between the privacy preservation and training time, which are in full agreement with our theory.

Our results apply to a wide spectrum of machine learning algorithms. This paper applies them to the iterative Gaussian mechanism with mini-batch (IGMM), which includes stochastic gradient Langevin dynamics [Welling and Teh, 2011] as an example of the stochastic gradient Markov chain Monte Carlo scheme [Ma et al., 2015] and agnostic federated learning [Geyer et al., 2017]. Our results deliver generalization bounds for agnostic federated learning. The obtained generalization bounds do not explicitly rely on the model size, which can be prohibitively large in modern methods, such as deep neural networks.

2 NOTATIONS AND PRELIMINARIES

Suppose $S = \{(x_1, y_1), \dots, (x_N, y_N) | x_i \in \mathcal{X} \subset \mathbb{R}^{d_X}, y_i \in \mathcal{Y} \subset \mathbb{R}^{d_Y}, i = 1, \dots, N\}$ is a training sample set, where x_i is the i -th feature, y_i is the corresponding label, and d_X and d_Y are the dimensions of the feature and the label, respectively. For the brevity, we define $z_i = (x_i, y_i)$. We also define random variables $Z = (X, Y)$, such that all $z_i = (x_i, y_i)$ are independent and identically distributed (i.i.d.) observations of the variable $Z = (X, Y) \in \mathcal{Z}$, $Z \sim \mathcal{D}$, where \mathcal{D} is the data distribution. For any set U , we denote its boundary points as ∂U . For any function $f(x) : \mathbb{R} \rightarrow \mathbb{R}$, we write $g = \tilde{\mathcal{O}}(f)$, if there exists another function h , such that $g = fh$, and for any $\alpha > 0$, $\lim_{x \rightarrow \infty} \frac{h(x)}{x^\alpha} = 0$. We also write $g = \tilde{\Omega}(f)$ if $f = \tilde{\mathcal{O}}(g)$, and $g = \tilde{\Theta}(f)$ if both $f = \tilde{\mathcal{O}}(g)$ and $g = \tilde{\mathcal{O}}(f)$. We also use \mathbf{p} as the probability density, with \mathbf{p}^V and \mathbb{P}^V respectively the probability density and the probability conditional on any random variable V .

A machine learning algorithm \mathcal{A} learns a hypothesis,

$$\mathcal{A}(S) \in \mathcal{H} \subset \mathcal{Y}^{\mathcal{X}} = \{f : \mathcal{X} \rightarrow \mathcal{Y}\},$$

from the training sample $S \in \mathcal{Z}^N$. The expected risk $\mathcal{R}_S(\mathcal{A}(S))$ and empirical risk $\hat{\mathcal{R}}_{\mathcal{D}}(\mathcal{A}(S))$ of the algorithm \mathcal{A} are defined as follows,

$$\begin{aligned} \mathcal{R}_{\mathcal{D}}(\mathcal{A}(S)) &= \mathbb{E}_{z \sim \mathcal{D}} \ell(\mathcal{A}(S), z), \\ \hat{\mathcal{R}}_S(\mathcal{A}(S)) &= \frac{1}{N} \sum_{i=1}^N \ell(\mathcal{A}(S), z_i), \end{aligned}$$

where $\ell : \mathcal{H} \times \mathcal{Z} \rightarrow \mathbb{R}^+$ is the loss function. It is worth noting that both the algorithm \mathcal{A} and the training sample set S can introduce randomness in the expected risk $\mathcal{R}_{\mathcal{D}}(\mathcal{A}(S))$ and empirical risk $\hat{\mathcal{R}}_S(\mathcal{A}(S))$. The generalization error is

defined as the difference between the expected risk and empirical risk,

$$\text{Gen}_{S, \mathcal{A}(S)} \triangleq \mathcal{R}_{\mathcal{D}}(\mathcal{A}(S)) - \hat{\mathcal{R}}_S(\mathcal{A}(S)),$$

whose upper bound is called the generalization bound.

Differential privacy measures the ability to preserve privacy, which is defined as follows (cf. [Dwork and Roth, 2014]).

Definition 1 (Differential Privacy). *A stochastic algorithm \mathcal{A} is called (ε, δ) -differentially private if for any hypothesis subset $\mathcal{H}_0 \subset \mathcal{H}$ and any neighboring sample set pair S and S' which differ by only one example (called S and S' adjacent), we have*

$$\log \left[\frac{\mathbb{P}_{\mathcal{A}(S)}(\mathcal{A}(S) \in \mathcal{H}_0) - \delta}{\mathbb{P}_{\mathcal{A}(S')}(\mathcal{A}(S') \in \mathcal{H}_0)} \right] \leq \varepsilon. \quad (2)$$

The algorithm \mathcal{A} is also called ε -differentially private, if it is $(\varepsilon, 0)$ -differentially private.

Differential privacy measures the "worst case" distance between the hypothesis distributions, in the sense that for a (ε, δ) differential privacy preserving algorithm, ε needs to be larger than the left hand of eq.(2) for any \mathcal{H}_0 . We then introduce KL divergence, which measures the "average" distance between two distributions, and are helpful in approximating differential privacy (cf. [Kullback and Leibler, 1951]).

Definition 2 (KL Divergence). *Suppose two distributions $P(\cdot)$ and $Q(\cdot)$ are defined on the same support \mathcal{H} . Then the KL divergence between Q and P is defined as*

$$D_{KL}(Q \| P) = \int_{h \in \mathcal{H}} \left(\log \frac{dQ(h)}{dP(h)} \right) dQ(h).$$

In this paper, we will slightly abuse the notations of distribution and its cumulative distribution function when no ambiguity is introduced, since there is a one-one mapping between them if zero-probability events are ignored.

3 GENERALIZATION BOUNDS FOR ITERATIVE DIFFERENTIALLY PRIVATE ALGORITHMS

This section establishes the generalizability of iterative differentially private algorithms. The establishment has two steps. We first establish generalization bounds for any differentially private learning algorithm. Then, we investigate how the iterative nature shared by most learning algorithms would influence the differential privacy and further the generalizability via three composition theorems. We also sketch the proofs for these results and demonstrate their advantages compared with existing literature.

3.1 BRIDGING GENERALIZATION AND PRIVACY PRESERVATION

We first prove a high-probability generalization bound for any (ε, δ) -differentially private machine learning algorithm as follows.

Theorem 1 (High-Probability Generalization Bound). *Suppose algorithm \mathcal{A} is (ε, δ) -differentially private, the training sample size (c_1, \dots, c_7) are some positive constants*

$$N \geq \max \left\{ \frac{c_1}{\varepsilon^2} \ln \left(\frac{c_2}{e^{-c_3 \varepsilon \delta}} \right), \frac{c_4}{c_5(1 - c_6 e^{-\varepsilon})} \ln c_7 e^{-\varepsilon \delta} \right\},$$

and the loss function $\|\ell\|_{\infty} \leq 1$. Then, for any data distribution \mathcal{D} over data space \mathcal{Z} , eq.(1) holds with

$$\begin{cases} a = 4\varepsilon, b = \frac{2e^{-1.7\varepsilon\delta}}{\varepsilon} \ln \left(\frac{2}{\varepsilon} \right) & , \varepsilon \leq \frac{1}{5}; \\ a = 1.2(1 - 0.9e^{-\varepsilon}), b = \frac{18e^{-\varepsilon\delta}}{1 - 0.9e^{-\varepsilon}} \ln \left(\frac{220}{1 - 0.9e^{-\varepsilon}} \right) & , \varepsilon > \frac{1}{5}. \end{cases}$$

Theorem 1 demonstrates that a good privacy-preserving ability implies a good generalizability. Thus, we can unify the algorithm designing for enhancing privacy preservation and for improving generalization.

3.1.1 Proof Sketch

We now give the proof sketch for Theorem 1 (the details are deferred to Appendix A). The proofs have three stages: (1) we first prove an on-average generalization bound for multi-sample-set learning algorithms defined as below; (2) we then obtain a high-probability generalization bound for multi-sample-set algorithms; and (3) we eventually prove Theorem 1 by reduction to absurdity.

Definition 3 (Multi-Sample-Set Learning Algorithms). *Suppose the training sample set S with size kN is separated to k sub-sample-sets S_1, \dots, S_k , each of which has the size of N . In another word, S is formed by k sub-sample-sets as*

$$S = (S_1, \dots, S_k).$$

The hypothesis $\mathcal{B}(S)$ learned by multi-sample-set algorithm \mathcal{B} on dataset S is defined as follows,

$$\mathcal{B} : \mathcal{Z}^{k \times N} \mapsto \mathcal{H} \times \{1, \dots, k\}, \mathcal{B}(S) = (h_{\mathcal{B}(S)}, i_{\mathcal{B}(S)}).$$

To obtain the high-probability generalization bound for a differentially private algorithm \mathcal{A} on N examples, we sample kN examples and divide them into k sub-sample-sets. A $(2\varepsilon, \delta)$ -differentially private multi-sample-set algorithm \mathcal{B} is constructed as (1) employing an (ε, δ) -differentially private algorithm \mathcal{A} to learn k hypotheses from the k sub-sample-sets; and (2) performing an ε -differentially private mechanism (e.g., exponential mechanism) to select the final hypothesis and its index (see Sec A.2). The generalization

bound of algorithm \mathcal{A} is then obtained from the bound of multi-sample-set algorithm \mathcal{B} (see Lemma 1).

Stage 1: Prove an on-average generalization bound for multi-sample-set learning algorithms.

We first prove the following on-average generalization bound for multi-sample-set learning algorithms.

Theorem 2 (On-Average Multi-Sample-Set Generalization Bound). *Let multi-sample-set algorithm $\mathcal{B} : \mathcal{Z}^{k \times N} \mapsto \mathcal{H} \times \{1, \dots, k\}$ be (ε, δ) -differentially private and the loss function $\|\ell\|_\infty \leq 1$. Then, for any data distribution \mathcal{D} over data space \mathcal{Z} , we have the following inequality,*

$$\left| \mathbb{E}_{S \sim \mathcal{D}^N, \mathcal{B}(S)} \left[\text{Gen}_{S, \mathcal{B}(S)} \right] \right| \leq e^{-\varepsilon} k \delta + 1 - e^{-\varepsilon}. \quad (3)$$

Stage 2: Prove a high-probability generalization bound for multi-sample-set algorithms.

Markov bound (cf. [Mohri et al., 2018], Theorem C.1) is an important concentration inequality in learning theory. Here, we slightly modify the original version as follows,

$$\mathbb{E}_x [h(x)] \geq \mathbb{E}_x [h(x) \mathbb{I}_{h(x) \geq g(x)}] \geq \mathbb{E}_x [g(x) \mathbb{I}_{h(x) \geq g(x)}].$$

Then, combining it with Theorem 2, we derive the following high-probability generalization bound for multi-sample-set algorithms.

Theorem 3 (High-Probability Multi-Sample-Set Generalization Bound). *Let all the notations be as Theorem 2. Then, if $\varepsilon \leq \frac{17}{50}$, for any $k \leq \frac{\varepsilon}{1.7e^{-1.7\varepsilon}\delta}$, we have*

$$\mathbb{P} \left(\text{Gen}_{S, \mathcal{B}(S), h_{\mathcal{B}(S)}} \leq ke^{-1.7\varepsilon}\delta + 1.7\varepsilon \right) \geq \frac{85\varepsilon}{127}.$$

Otherwise, if $\varepsilon > \frac{17}{50}$, for any $k \leq \frac{1-e^{-\varepsilon}}{10e^{-\varepsilon}\delta}$,

$$\mathbb{P} \left(\text{Gen}_{S, \mathcal{B}(S), h_{\mathcal{B}(S)}} \leq ke^{-\varepsilon}\delta + 1.1(1 - e^{-\varepsilon}) \right) \geq \frac{1 - e^{-\varepsilon}}{219}.$$

Stage 3: Prove Theorem 1 by Reduction to Absurdity.

We eventually prove Theorem 1 by *reduction to absurdity*. Assume there exists an algorithm \mathcal{A} which conflicts with Theorem 1. We can then construct an algorithm \mathcal{B} based on the exponential mechanism which is defined as follows (cf. [McSherry and Talwar, 2007]).

Definition 4 (Exponential Mechanism). *Suppose that S is a sample set, \mathcal{I} is an index set, ε is the privacy parameter, and $u : (S, r) \mapsto \mathbb{R}^+$ is a function with the sensitivity Δu defined by*

$$\Delta u \triangleq \max_{i \in \mathcal{I}} \max_{S, S' \text{ adjacent}} |u(S, i) - u(S', i)|.$$

Then, the exponential mechanism $\mathcal{E}(S, u, \mathcal{I}, \varepsilon)$ outputs an element $i \in \mathcal{I}$ with probability proportional to $\exp(\frac{\varepsilon u(S, i)}{2\Delta u})$.

Then, we can prove the following lemma.

Lemma 1. *Suppose a positive integer N satisfies that $N \geq \max\{\frac{2}{0.077\varepsilon^2} \ln(\frac{43}{254e^{-1.7\varepsilon}\delta}), \frac{200}{\ln 0.9(1-0.9e^{-\varepsilon})} \ln \frac{9e^{-\varepsilon}\delta}{48180}\}$. If for an (ε, δ) differential privacy preserving algorithm $\mathcal{A} : \mathcal{Z}^N \rightarrow \mathcal{H}$, there is*

$$\mathbb{P}(\text{Gen}_{S, \mathcal{A}(S)} \leq a) < \frac{b}{2}, \quad (4)$$

for a and b defined as Theorem 1. Then if $\varepsilon < \frac{1}{5}$, there exists a multi-sample-set algorithm $\mathcal{B} : \mathcal{Z}^{k \times N} \rightarrow \mathcal{H} \times \{1, \dots, k\}$ with $(1.7\varepsilon, \delta)$ -differential privacy, $k = \frac{\varepsilon}{e^{-1.7\varepsilon}\delta}$, and

$$\mathbb{P} \left(\text{Gen}_{\tilde{S}, \mathcal{B}(\tilde{S}), h_{\mathcal{B}(\tilde{S})}} \leq ke^{-1.7\varepsilon}\delta + 1.7 \times 1.7\varepsilon \right) < \frac{85\varepsilon}{127}. \quad (5)$$

If $\varepsilon \geq \frac{1}{5}$, there exists a multi-sample-set algorithm $\mathcal{B} : \mathcal{Z}^{k \times N} \rightarrow \mathcal{H} \times \{1, \dots, k\}$ with $(\varepsilon - \ln(0.9), \delta)$ -differential privacy, $k = \frac{1-0.9e^{-\varepsilon}}{9e^{-\varepsilon}\delta}$, and

$$\begin{aligned} & \mathbb{P} \left(\text{Gen}_{\tilde{S}, \mathcal{B}(\tilde{S}), h_{\mathcal{B}(\tilde{S})}} \leq 0.9ke^{-\varepsilon}\delta + 1.1(1 - 0.9e^{-\varepsilon}) \right) \\ & < \frac{1 - 0.9e^{-\varepsilon}}{219}. \end{aligned} \quad (6)$$

Therefore, for any algorithm \mathcal{A} on which Theorem 1 fails to hold, eq. (5) or eq.(6) in Lemma 1 will conflict with Theorem 3, which completes the proof of Theorem 1.

Remark 1. *Obtained bounds are all in the same form as eq.(1), where multi-sample-set algorithms help significantly decrease b with a small increase of a . In this way, we obtain a tighter generalization bound.*

3.1.2 Comparison with Existing Results

This section compares our results with the existing works.

Comparison of Theorem 1. There have been several high-probability generalization bounds for (ε, δ) -differentially private machine learning algorithms.

Dwork et al. [2015] proved that

$$\mathbb{P}[\text{Gen}_{S, \mathcal{A}(S)} < 4\varepsilon] > 1 - 8\delta^\varepsilon.$$

Oneto et al. [2017] proved that

$$\begin{aligned} & \mathbb{P} \left[\text{Gen}_{S, \mathcal{A}(S)} < \sqrt{6\hat{\mathcal{R}}_S(\mathcal{A}(S))\hat{\varepsilon}} + 6\varepsilon^2 + \frac{6}{N} \right] \\ & > 1 - 3e^{-N\varepsilon^2}, \\ & \mathbb{P} \left[\text{Gen}_{S, \mathcal{A}(S)} < \sqrt{4\hat{V}_S(\mathcal{A}(S))\hat{\varepsilon}} + \frac{5N}{N-1}(\varepsilon^2 + 1/N) \right] \\ & > 1 - 3e^{-N\varepsilon^2}, \end{aligned}$$

where $\hat{\varepsilon} = \varepsilon + \sqrt{1/N}$, and $\hat{V}_S(\mathcal{A}(S))$ is the empirical variance of $\ell(\mathcal{A}(S), \cdot)$:

$$\hat{V}_S(\mathcal{A}(S)) = \text{Cov}_{z \sim S}(\ell(\mathcal{A}(S), z)).$$

Nissim and Stemmer [2015] proved that

$$\mathbb{P} [|\text{Gen}_{S, \mathcal{A}(S)}| < 13\varepsilon] > 1 - \frac{2\delta}{\varepsilon} \log \left(\frac{2}{\varepsilon} \right).$$

This is the existing tightest high-probability generalization bound in the literature. However, this bound only stands for binary classification problems with $\varepsilon \leq \frac{1}{10}$. By contrast, our high-probability generalization bound holds for any machine learning algorithm and stays valid for any $\varepsilon > 0$.

Also, our bound is strictly tighter. All the bounds, including ours, are in the same form as eq.(1). Apparently, a smaller a and a smaller b imply a tighter generalization bound. Our bound improves the current tightest result from two aspects:

- For $\varepsilon \leq \frac{1}{10}$, our bounds tightens the term a from 13ε to 4ε .
- For $\varepsilon \leq \frac{1}{10}$, our bounds tightens the term b from $\frac{2\delta}{\varepsilon} \log \left(\frac{2}{\varepsilon} \right)$ to $\frac{2e^{-1.7\varepsilon}\delta}{\varepsilon} \log \left(\frac{2}{\varepsilon} \right)$.
- Our bounds further cover the case when $\varepsilon > \frac{1}{10}$.

These improvements are significant. Abadi et al. [2016] conducted experiments on the differential privacy in deep learning. Their empirical results demonstrate that the factor ε can be as large as 10.

Comparison of Theorem 2. There is only one related work in the literature that presents an on-average generalization bound for multi-sample-set algorithm. Nissim and Stemmer [2015] proved that,

$$\left| \mathbb{E}_{S \sim \mathcal{D}^N, \mathcal{B}(S)} \left[\text{Gen}_{S, i_{\mathcal{B}(S)}, h_{\mathcal{B}(S)}} \right] \right| \leq k\delta + 2\varepsilon.$$

Our bound is tighter by a factor of $\frac{e^\varepsilon}{2}$. According to the empirical results by [Abadi et al., 2016], this factor can be as large as $e^{10} \approx 20,000$. It is a significant multiplier for loss function. Furthermore, the result by [Nissim and Stemmer, 2015] stands only for binary classification, while our result applies to all differentially private learning algorithms.

3.2 HOW THE ITERATIVE NATURE CONTRIBUTES?

Most machine learning algorithms are iterative, which may degenerate the privacy-preserving ability along with iterations. This section studies the degenerative nature of the privacy preservation in iterative machine learning algorithms and its influence to the generalization.

We have the following composition theorem.

Theorem 4 (Composition Theorem I). *Suppose an iterative machine learning algorithm \mathcal{A} has T steps: $\{W_i(S)\}_{i=0}^T$, where W_i is the learned hypothesis after the i -th iteration. Suppose the i -th iterator*

$$\mathcal{M}_i : (W_{i-1}, S) \mapsto W_i$$

is (ε, δ) -differentially private. Then, the algorithm \mathcal{A} is (ε', δ') -differentially private, where $\varepsilon' = \min \{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3\}$ with

$$\begin{aligned} \varepsilon'_1 &= \sum_{i=1}^T \varepsilon_i, \\ \varepsilon'_2 &= \sqrt{2 \sum_{i=1}^T \varepsilon_i^2 \log \left(e + \frac{\sqrt{\sum_{i=1}^T \varepsilon_i^2}}{\tilde{\delta}} \right)} \\ &\quad + \sum_{i=1}^T \frac{(e^{\varepsilon_i} - 1) \varepsilon_i}{e^{\varepsilon_i} + 1}, \\ \varepsilon'_3 &= \sum_{i=1}^T \frac{(e^{\varepsilon_i} - 1) \varepsilon_i}{e^{\varepsilon_i} + 1} + \sqrt{2 \log \left(\frac{1}{\tilde{\delta}} \right) \sum_{i=1}^T \varepsilon_i^2}, \end{aligned}$$

and $\tilde{\delta}$ is an arbitrary positive real constant.

Correspondingly, the factor δ' is defined as the maximal value of the following equation with respect to $\{\alpha_i\}_{i=1}^T \in I$,

$$2 - \prod_{i=1}^T \left(1 - e^{\alpha_i} \frac{\delta_i}{1 + e^{\varepsilon_i}} \right) - \prod_{i=1}^T \left(1 - \frac{\delta_i}{1 + e^{\varepsilon_i}} \right) + \tilde{\delta}, \quad (7)$$

where $I = \partial \{ \sum_{i=1}^T \alpha_i = \varepsilon', \varepsilon_i \geq \alpha_i \geq 0 \}$, and $\tilde{\delta}$ is the same real constant mentioned above.

When all the iterations have the same privacy-preserving ability, we can tighten the approximation of the factor δ' as the following corollary.

Corollary 1 (Composition Theorem II). *When all the iterations are (ε, δ) -differential private, δ' is*

$$\begin{aligned} \delta' &= 1 - \left(1 - e^\varepsilon \frac{\delta}{1 + e^\varepsilon} \right)^{\lceil \frac{\varepsilon'}{\varepsilon} \rceil} \left(1 - \frac{\delta}{1 + e^\varepsilon} \right)^{T - \lceil \frac{\varepsilon'}{\varepsilon} \rceil} \\ &\quad + 1 - \left(1 - \frac{\delta}{1 + e^\varepsilon} \right)^T + \tilde{\delta} \\ &= \left(T - \left\lceil \frac{\varepsilon'}{\varepsilon} \right\rceil \right) \frac{2\delta}{1 + e^\varepsilon} + \left\lceil \frac{\varepsilon'}{\varepsilon} \right\rceil \delta + \tilde{\delta} + \mathcal{O} \left(\left(\frac{\delta}{1 + e^\varepsilon} \right)^2 \right). \end{aligned}$$

Here we make some explanation about the above corollary. The maximum of δ' is achieved when at most $T - \left\lceil \frac{\varepsilon'}{\varepsilon} \right\rceil$ elements $\alpha_i \neq 0$. We note that

$$(1 - x)^n = 1 - nx + \mathcal{O}(x^2).$$

Then, the δ' in Theorem 4 can be estimated as

$$\begin{aligned} \delta' &= 1 - \left(1 - e^\varepsilon \frac{\delta}{1 + e^\varepsilon}\right)^{\lceil \frac{\varepsilon'}{\varepsilon} \rceil} \left(1 - \frac{\delta}{1 + e^\varepsilon}\right)^{T - \lceil \frac{\varepsilon'}{\varepsilon} \rceil} \\ &\quad + 1 - \left(1 - \frac{\delta}{1 + e^\varepsilon}\right)^T + \tilde{\delta} \\ &\approx \left(T - \lceil \frac{\varepsilon'}{\varepsilon} \rceil\right) \frac{2\delta}{1 + e^\varepsilon} + \lceil \frac{\varepsilon'}{\varepsilon} \rceil \delta + \tilde{\delta}. \end{aligned}$$

When all the iterators \mathcal{M}_i satisfy $\varepsilon_i \equiv \varepsilon$, we can further tighten the estimation of δ' for ε'_3 in Theorem 4.

Corollary 2 (Composition Theorem III). *Suppose the iterators \mathcal{M}_i are (ε, δ_i) -differentially private and all the other conditions in Theorem 4 hold. Then, algorithm \mathcal{A} is (ε', δ') -differentially private, where $\varepsilon' = \min\{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3\}$, and*

$$\begin{aligned} \delta' &= e^{-\frac{\varepsilon' + T\varepsilon}{2}} \left(\frac{1}{1 + e^\varepsilon} \left(\frac{2T\varepsilon}{T\varepsilon - \varepsilon'}\right)\right)^T \left(\frac{T\varepsilon + \varepsilon'}{T\varepsilon - \varepsilon'}\right)^{-\frac{\varepsilon' + T\varepsilon}{2\varepsilon}} \\ &\quad + 2 - \left(1 - e^\varepsilon \frac{\delta}{1 + e^\varepsilon}\right)^{\lceil \frac{\varepsilon'}{\varepsilon} \rceil} \left(1 - \frac{\delta}{1 + e^\varepsilon}\right)^{T - \lceil \frac{\varepsilon'}{\varepsilon} \rceil} \\ &\quad - \left(1 - \frac{\delta}{1 + e^\varepsilon}\right)^T. \end{aligned}$$

Furthermore, for the case $\varepsilon' = \varepsilon'_3$, δ' is strictly tighter than that in Theorem 4.

The three composition theorems extend the developed relationship between generalization and privacy preservation to iterative machine learning algorithms. At this point, we establish the theoretical foundation for the generalizability of iterative differentially private machine learning algorithms.

3.2.1 Proof Sketch

We now sketch the proofs for Theorem 4 (for more details, please refer to Appendix B). The proofs have four stages: (1) we first approximate the KL-divergence between hypotheses learned on neighboring training sample sets; (2) we then prove a composition bound for algorithms with step i ε_i -differentially private; (3) this composition theorem is extended to for algorithms with step i $(\varepsilon_i, \delta_i)$ -differentially private; and (4) we eventually tighten the result in (3) to obtain Theorem 4. We also proved two additional composition theorems as by-products. The two composition theorems are weaker than Theorem 4 but play essential roles in the proofs.

It is worth noting that the proofs of composition theorems are significantly different from [Kairouz et al., 2017] which calculate the composition combining ε and δ . This combination causes difficulties in obtaining tight estimation. In contrast, our Theorem 4 and Corollary 2 separate the considerations on ε and δ by coupling technique and momentum method, which considerably improve the composition theorem.

Stage 1: Approximate the KL-divergence between hypotheses learned on neighboring training sample sets.

It would be technically difficult to approach directly the differential privacy of an iterative learning algorithm from the differential privacy of every iteration. To relieve the technical difficulty, we employ KL divergence as a bridge in this paper. For any ε -differentially private learning algorithm, we prove the following lemma to approximate the KL-divergence between hypotheses learned on neighboring training sample sets.

Lemma 2. *If \mathcal{A} is an ε -differentially private algorithm, then for every neighbor database pair S and S' , the KL divergence between hypotheses $\mathcal{A}(S)$ and $\mathcal{A}(S')$ satisfies the following inequality,*

$$D_{KL}(\mathcal{A}(S) \parallel \mathcal{A}(S')) \leq \varepsilon \frac{e^\varepsilon - 1}{e^\varepsilon + 1}.$$

To the best of our knowledge, Lemma 2 is the current tightest bound of KL divergence by differential privacy parameter ε . There are two related results in the literature, which are considerably looser than ours. Dwork et al. [2010] proved an inequality of the KL divergence as follows,

$$D_{KL}(\mathcal{A}(S) \parallel \mathcal{A}(S')) \leq \varepsilon(e^\varepsilon - 1).$$

Then, Dwork and Rothblum [2016] further improved it to

$$D_{KL}(\mathcal{A}(S) \parallel \mathcal{A}(S')) \leq \frac{1}{2}\varepsilon(e^\varepsilon - 1). \quad (8)$$

Compared with ours, eq. (8) is larger by a factor $(1 + e^\varepsilon)/2$, which can be very large in practice.

Stage 2: Prove a weaker composition theorem where the i th iteration is ε_i -differential private.

Based on Lemma 2, we can prove the following composition theorem as a preparation theorem.

Theorem 5 (Composition Theorem IV). *Suppose an iterative machine learning algorithm \mathcal{A} has T steps: $\{W_i(S)\}_{i=1}^T$. Specifically, we define the i -th iterator as follows,*

$$\mathcal{M}_i : (W_{i-1}(S), S) \mapsto W_i(S). \quad (9)$$

Assume that W_0 is the initial hypothesis (which does not depend on S). If for any fixed W_{i-1} , $\mathcal{M}_i(W_{i-1}, S)$ is ε_i -differentially private, then $\{W_i\}_{i=0}^T$ is (ε', δ') -differentially private that

$$\varepsilon' = \sqrt{2 \log \left(\frac{1}{\delta'}\right) \left(\sum_{i=1}^T \varepsilon_i^2\right) + \sum_{i=1}^T \varepsilon_i \frac{e^{\varepsilon_i} - 1}{e^{\varepsilon_i} + 1}}.$$

Stage 2: Prove a weaker composition theorem where the i th iteration is $(\varepsilon_i, \delta_i)$ -differentially private. The following technical lemma is adopted to derive the generalization bound:

Lemma 3 (cf. [Dwork and Roth, 2014], Theorem 3.17). *For any random variables Y and Z , we have that*

$$D_\infty^\delta(Y\|Z) \leq \varepsilon, D_\infty^\delta(Z\|Y) \leq \varepsilon,$$

if and only if there exist random variables Y', Z' such that

$$\Delta(Y\|Y') \leq \frac{\delta}{e^\varepsilon + 1}, \Delta(Z\|Z') \leq \frac{\delta}{e^\varepsilon + 1},$$

$$D_\infty(Y'\|Z') \leq \varepsilon, D_\infty(Z'\|Y') \leq \varepsilon.$$

Here D_∞^δ is the δ -approximate max divergence, D_∞ is the max divergence, and Δ is the statistical distance, and we defer the formal definitions to Appendix B.1.

Intuitively, lemma 3 allows us to separate ε and δ for variables with δ max divergence ε and derive coupled variables with max divergence ε . Based on Lemmas 3, we can prove the following composition theorem whose estimate of ε' is somewhat looser than our main results.

Theorem 6 (Composition Theorem V). *Let all the conditions and notations in Theorem 5 hold except that the i th step is $(\varepsilon_i, \delta_i)$ differentially private. Then, $\{W_i\}_{i=0}^T$ is (ε', δ') -differentially private where*

$$\varepsilon' = \sqrt{2 \log \left(\frac{1}{\tilde{\delta}} \right) \left(\sum_{i=1}^T \varepsilon_i^2 \right) + \sum_{i=1}^T \varepsilon_i \frac{e^{\varepsilon_i} - 1}{e^{\varepsilon_i} + 1}},$$

$$\delta' = \max_{\{\alpha_i\}_{i=1}^T \in \partial I} \tilde{\delta} + 2 - \prod_{i=1}^T \left(1 - e^{\alpha_i} \frac{\delta_i}{1 + e^{\varepsilon_i}} \right) - \prod_{i=1}^T \left(1 - \frac{\delta_i}{1 + e^{\varepsilon_i}} \right),$$

$$\text{and } I = \left\{ \sum_{i=1}^T \alpha_i = \varepsilon', \alpha_i \geq 0 \right\}.$$

Stage 4: Prove Theorem 4.

By the same routine of deriving Theorem 6 together with Theorem 3.5 in [Kairouz et al., 2017], we eventually extend the weaker versions to Theorem 4.

3.2.2 Comparison with Existing Results

Our composition theorem is strictly tighter than the existing results: a classic composition theorem is as follows (see [Dwork and Roth, 2014], Theorem 3.20 and Corollary 3.21, pp. 49-52),

$$\varepsilon' = \sum_{i=1}^T \varepsilon_i (e^{\varepsilon_i} - 1) + \sqrt{2 \log \left(\frac{1}{\tilde{\delta}} \right) \sum_{i=1}^T \varepsilon_i^2},$$

$$\delta' = \tilde{\delta} + \sum_{i=1}^T \delta_i,$$

where $\tilde{\delta}$ is an arbitrary positive real number, (ε', δ') is the differential privacy of the whole algorithm, and $(\varepsilon_i, \delta_i)$ is the differential privacy of the i -th iteration.

Currently, the tightest approximation is given by [Kairouz et al., 2017] as follows,

$$\varepsilon' = \min \{ \varepsilon'_1, \varepsilon'_2, \varepsilon'_3 \},$$

$$\delta' = 1 - (1 - \delta)^T (1 - \tilde{\delta}),$$

where $\varepsilon'_1, \varepsilon'_2$, and ε'_3 are the same as those in Theorem 4. Therefore, their estimate of the ε' is the same as ours, while their δ' is also larger than ours approximately by

$$\delta \frac{e^\varepsilon - 1}{e^\varepsilon + 1} \left(T - \left\lceil \frac{\varepsilon'}{\varepsilon} \right\rceil \right).$$

The iteration number T is usually overwhelmingly large, which guarantees our advantage is significant.

4 EXPERIMENTS

We conducted a systematic experiment to verify three relationships suggested by our theory: (1) a positive correlation between privacy preservation and generalization; (2) a negative correlation between privacy preservation and training time; and (3) a negative correlation between generalization and training time. The empirical results are in full agreement with the theoretical results.

4.1 IMPLEMENTATION DETAILS

Three benchmark image datasets, MNIST [LeCun, 1998], CIFAR-10 and CIFAR-100 [Krizhevsky et al., 2009], are used in our experiments. The separations of the training sets and the test sets are the same as the official versions. For MNIST, we trained one-hidden-layer MLP, in which the number of the hidden neurons is 100. For CIFAR-10 and CIFAR-100, we trained VGG-16 [Simonyan and Zisserman, 2014] and ResNet-18 [He et al., 2016].

SGD is employed to train the model. The batch size is set as 128, the momentum factor is set as 0.9, and the weight decay factor is set as 0.0005. For the experiments on MNIST, every model is trained for 50 epochs. The learning rate is initialized as 0.1 and decayed by 0.1 every 20 epochs. For the experiments on CIFAR-10 and CIFAR-100, every model is trained for 100 epochs. The learning rate is initialized as 0.1 and decays by 0.1 every 40 epochs.

Membership inference attack [Yeom et al., 2018, Shokri et al., 2017] is performed to the model in every epoch in order to evaluate the privacy-preserving ability. Usually, a larger membership inference attack indicates a weaker privacy-preserving ability. The implementation of membership inference attack is from JD Explore Academy Trustworthy AI Toolkit. We also calculate the generalization error (the difference between training and test errors) of the model in every epoch. Similarly, a larger generalization error indicates a weaker generalizability.

Table 1: Spearman’s rank order correlation coefficient (SCC) and p value of: (1) membership inference attack accuracy and generalization error; (2) membership inference attack accuracy and training time; and (3) generalization error and training time.

	MLP - MNIST		VGG-16 - CIFAR-10		ResNet-18 - CIFAR-10		VGG-16 - CIFAR-100		ResNet-18 - CIFAR-100	
	SCC	p	SCC	p	SCC	p	SCC	p	SCC	p
Privacy - Gen. Err.	0.92	1.2×10^{-21}	0.92	1.8×10^{-41}	0.80	1.1×10^{-23}	1.00	6.4×10^{-107}	0.82	4.2×10^{-25}
Privacy - Time	0.96	1.7×10^{-27}	0.98	9.9×10^{-73}	0.97	1.5×10^{-64}	0.99	5.0×10^{-91}	0.98	8.8×10^{-72}
Gen. Err. - Time	0.91	8.6×10^{-20}	0.91	2.6×10^{-40}	0.80	2.1×10^{-23}	0.99	5.5×10^{-87}	0.82	7.0×10^{-26}

Algorithm 1 Iterative Gaussian Mechanism with Mini-batch (IGMM)

Require: Sample $S = \{z_1, \dots, z_N\}$, Gaussian noise variance σ , size of mini-batch τ , iteration steps T , learning rate $\{\eta_1, \dots, \eta_T\}$, update function $g(z, W)$, and its diameter $D \triangleq \max_{W, z, z'} \|g(z, W) - g(z', W)\|$.

- 1: Initialize W_0 randomly.
 - 2: For $t = 1$ to T do:
 - 3: Uniformly sample a mini-batch \mathcal{B}_t of size τ from S without replacement;
 - 4: Sample g_t from $\sigma\mathcal{N}(0, \mathbb{I})$;
 - 5: Update $W_t \leftarrow W_{t-1} - \eta_t \left[\frac{1}{\tau} \sum_{z \in \mathcal{B}_t} g(z, W_{t-1}) + g_t \right]$.
-

The collected (membership inference attack accuracy, training time) pairs, (generalization error, training time) pairs, and (membership inference attack accuracy, generalization error) pairs are divided into groups according to neural network architectures and datasets. Spearman’s rank-order correlation test [Spearman, 1987] is performed on every group.

4.2 EMPIRICAL RESULTS

The Spearman’s rank-order correlation coefficients and the p values are collected in Table 1. The p values are all smaller than 0.05, which show the following three correlations are statistically significant: (1) a positive correlation between privacy preservation and generalization; (2) a negative correlation between privacy preservation and training time; and (3) a negative correlation between generalization and training time. These results are in full agreement with our theory.

5 APPLICATIONS

Our theories apply to a wide spectrum of machine learning algorithms. This paper implements them to the popular iterative Gaussian mechanisms with mini-batch (1) as an example. Gaussian mechanism are widely adopted in privacy-preserving machine learning. In this paper, we show the application in agnostic federated learning.

We first formally describe iterative Gaussian mechanism with mini-batch in Algorithm 4.1. We then apply Theorem 1 and Corollary 2 to estimate the differential privacy and deliver a generalization bound for SGLD.

Theorem 7. *IGMM described as Algorithm 1 is (ϵ', δ') -differentially private, where*

$$\begin{aligned} \epsilon' &= \sqrt{2T \log\left(\frac{1}{\delta}\right)} \tilde{\epsilon}^2 + T \tilde{\epsilon} \frac{e^{\tilde{\epsilon}} - 1}{e^{\tilde{\epsilon}} + 1}, \\ \delta' &= 2 - \left(1 - \frac{\delta}{1 + e^{\tilde{\epsilon}}}\right)^T + \tilde{\delta}' \\ &\quad - \left(1 - \frac{\delta e^{\tilde{\epsilon}}}{1 + e^{\tilde{\epsilon}}}\right)^{\lceil \frac{N\epsilon'}{\tau\tilde{\epsilon}} \rceil} \left(1 - \frac{\delta}{1 + e^{\tilde{\epsilon}}}\right)^{T - \lceil \frac{N\epsilon'}{\tau\tilde{\epsilon}} \rceil}, \end{aligned}$$

$\tilde{\epsilon}$ is defined as

$$\tilde{\epsilon} = \log\left(\frac{N - \tau}{N} + \frac{\tau}{N} \exp\left(\frac{\sqrt{2}D\sigma\frac{1}{\tau}\sqrt{\log\frac{1}{\delta}} + \frac{1}{\tau^2}D^2}\right)\right),$$

and $\tilde{\delta}'$ is given as

$$\tilde{\delta}' = e^{-\frac{\epsilon' + T\tilde{\epsilon}}{2}} \left(\frac{T\tilde{\epsilon} + \epsilon'}{T\tilde{\epsilon} - \epsilon'}\right)^{-\frac{\epsilon' + T\tilde{\epsilon}}{2\tilde{\epsilon}}} \left(\frac{1}{1 + e^{\tilde{\epsilon}}}\right)^T \left(\frac{2T\tilde{\epsilon}}{T\tilde{\epsilon} - \epsilon'}\right)^T.$$

Additionally, a generalization bound is delivered by combining with Theorem 1.

Remark 2. *The generalization bound in Theorem 7 does not explicitly rely on the model size. One may argue the diameter R of update function g implicitly relies on the model size. However, in deep learning, g usually stays in a low rank manifold. As an example, Yu et al. [2021] empirically show that the linear space formed by gradients in deep learning is of low dimension, largely independent of the model size. This dismisses the possibility of introducing dependence on model size by the gradient distance.*

Agnostic federated learning is a special case of IGMM. Therefore, by Theorem 7, we have the following asymptotic high probability generalization bound for agnostic federated learning in terms of client number C .

Corollary 3 (Asymptotic generalization bound for agnostic federated learning). *For agnostic federated learning algorithm \mathcal{A}^{fed} with client set S with size C , the generalization error is bounded as*

$$\mathbb{P}\left(|\text{Gen}_{S, \mathcal{A}_S^{fed}}| > \tilde{\mathcal{O}}\left(\frac{1}{\sqrt{C}}\right)\right) < \tilde{\mathcal{O}}\left(\frac{1}{C^{\frac{3}{2}}}\right). \quad (10)$$

Remark 3. *The high probability generalization bound eq.(10) also holds for SGLD with N the sample size.*

As an over-parameterized model, deep learning has demonstrated excellent generalizability, which is somehow beyond the explanation of the existing statistical learning theory and thus attracts the community’s interest [E et al., 2020, He and Tao, 2020]. Recent advances include generalization bounds via hypothesis complexity [Bartlett et al., 2019, Golowich et al., 2018, Bartlett et al., 2017, Liang et al., 2019, Tu et al., 2020], PAC-Bayesian framework [Neyshabur et al., 2018], algorithmic stability [Hardt et al., 2016], and stochastic gradient descent or its variant [Mandt et al., 2017, Mou et al., 2018] on the loss surface [Kawaguchi, 2016, Yun et al., 2019, He et al., 2020]. A major difficulty in explaining deep learning’s excellent generalizability is that deep learning models usually has prohibitively large parameter size which make many generalization bounds vacuous.

Additionally, deep learning has become a promising player in many real-world applications, including financial services Fischer and Krauss [2018], healthcare Wang et al. [2012], and biometric authentication Snelick et al. [2005], in which the privacy-preserving ability is of vital importance. Several works have also studied the privacy preservation of deep learning and how to improve it further Abadi et al. [2016], Arachchige et al. [2019].

This work establishes generalization bounds for iterative learning algorithms via differential privacy, which do not explicitly rely on the model size. Our results also shed light to understanding the generalizability of deep learning from the privacy-preserving view.

6 CONCLUSION

This paper studies the relationships between generalization and privacy preservation in two steps. We first establish the relationship between generalization and privacy preservation for any machine learning algorithm. Specifically, we prove a high-probability bound for differentially private learning algorithms based on a novel on-average generalization bound for multi-sample-set algorithms. Then, we prove three composition theorems that calculate the (ϵ', δ') -differential privacy of an iterative algorithm. By integrating the two steps, we establish generalization guarantees for iterative differentially private machine learning algorithms. Compared with existing works, our theoretical results are strictly tighter and apply to a wider application domain. We then use them to the Gaussian mechanism. The obtained generalization bounds do not explicitly rely on the model size which would be prohibitively large in many modern models, such deep neural networks. The empirical results of MLP, VGG, and ResNet on the MNIST, CIFAR-10, and CIFAR-100 datasets are in full agreement with our theory.

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A PROOFS OF GENERALIZATION BOUNDS VIA DIFFERENTIAL PRIVACY

This appendix collects all the proofs of theorems and technical lemma in Section 3. It is organized as follows: (1) in Appendix A.1, we prove Theorem 2, Theorem 3, and Lemma 1, which are the preparations of the proof of Theorem 1; and (2) in Appendix A.2 we prove Theorem 1.

A.1 PROOF OF THEOREM 2, THEOREM 3, AND LEMMA 1

We first present the proof of Theorem 2.

Proof of Theorem 2. We first rewrite the expectation of the empirical risk of algorithm \mathcal{B} can be rewritten as

$$\begin{aligned}
& \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S, i_{\mathcal{B}(S)}} (h_{\mathcal{B}(S)}) \right] \right] = \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\mathbb{E}_{z \sim S_{i_{\mathcal{B}(S)}}} [\ell(h_{\mathcal{B}(S)}, z)] \right] \right] \\
& \stackrel{(*)}{=} \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\mathbb{E}_{\bar{z} \sim S} [\ell(h_{\mathcal{B}(S)}, z_{i_{\mathcal{B}(S)}})] \right] \right] = \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\bar{z} \sim S} \left[\mathbb{E}_{\mathcal{B}(S)} [\ell(h_{\mathcal{B}(S)}, z_{i_{\mathcal{B}(S)}})] \right] \right] \\
& = \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\bar{z} \sim S} \left[\mathbb{E}_{\mathcal{B}(S)} [\ell(h_{\mathcal{B}(S)}, z_{i_{\mathcal{B}(S)}})] \right] \right] \\
& = \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\bar{z} \sim S} \left[\int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S)}, z_{i_{\mathcal{B}(S)}}) \leq t) dt \right] \right] \\
& = \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\bar{z} \sim S} \left[\sum_{i=1}^k \int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S)}, z_i) \leq t, i_{\mathcal{B}(S)} = i) dt \right] \right],
\end{aligned}$$

where \bar{z} in the right side of eq.(*) is defined as $\{z_1, \dots, z_k\}$, and z_i is uniformly selected from S_i . Since \mathcal{B} is (ε, δ) -differentially private, we further have

$$\begin{aligned}
& \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\bar{z} \sim S} \left[\sum_{i=1}^k \int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S)}, z_i) \leq t, i_{\mathcal{B}(S)} = i) dt \right] \right] \\
& \leq \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\bar{z} \sim S, z_0 \sim D} \left[\sum_{i=1}^k \int_0^1 (e^\varepsilon \mathbb{P}(\ell(h_{\mathcal{B}(S^{z_i:z_0})}, z_i) \leq t, i_{\mathcal{B}(S^{z_i:z_0})} = i) + \delta) dt \right] \right] \\
& = e^\varepsilon \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\bar{z} \sim S, z_0 \sim D} \left[\sum_{i=1}^k \int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S^{z_i:z_0})}, z_i) \leq t, i_{\mathcal{B}(S^{z_i:z_0})} = i) dt \right] \right] + k\delta \\
& = \sum_{i=1}^k e^\varepsilon \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\bar{z} \sim S, z_0 \sim D} \left[\int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S^{z_i:z_0})}, z_i) \leq t, i_{\mathcal{B}(S^{z_i:z_0})} = i) dt \right] \right] + k\delta \\
& = \sum_{i=1}^k e^\varepsilon \mathbb{E}_{S' \sim \mathcal{D}^{kN-1}} \left[\mathbb{E}_{z_i \sim D, z_0 \sim D} \left[\int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S' \cup \{z_0\})}, z_i) \leq t, i_{\mathcal{B}(S' \cup \{z_0\})} = i) dt \right] \right] + k\delta.
\end{aligned}$$

Let $S = S' \cup \{z_0\}$ and $z = z_i$ (it is without loss of generality since all z_i is i.i.d. drawn from \mathcal{D}). Since $S' \cup \{z_0\} \sim \mathcal{D}^{kN}$, we have

$$\begin{aligned}
& \sum_{i=1}^k e^\varepsilon \mathbb{E}_{S' \sim \mathcal{D}^{kN-1}} \left[\mathbb{E}_{z_i \sim D, z_0 \sim D} \left[\int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S' \cup \{z_0\})}, z_i) \leq t, i_{\mathcal{B}(S' \cup \{z_0\})} = i) dt \right] \right] + k\delta \\
& = \sum_{i=1}^k e^\varepsilon \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{z \sim \mathcal{D}} \left[\int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S)}, z) \leq t, i_{\mathcal{B}(S)} = i) dt \right] \right] + k\delta \\
& = e^\varepsilon \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{z \sim \mathcal{D}} \left[\int_0^1 \mathbb{P}(\ell(h_{\mathcal{B}(S)}, z) \leq t) dt \right] \right] + k\delta \\
& = e^\varepsilon \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{z \sim \mathcal{D}} [\mathbb{E}_{\mathcal{B}(S)} [\ell(h_{\mathcal{B}(S)}, z)]] \right] + k\delta.
\end{aligned}$$

Therefore, we have

$$\mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right] \leq k\delta + e^\varepsilon \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} [\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)})] \right]. \quad (11)$$

Rearranging eq. (11), we have

$$\begin{aligned} e^{-\varepsilon} \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right] &\leq e^{-\varepsilon} k\delta + \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} [\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)})] \right] \\ &- \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} [\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)})] \right] \leq e^{-\varepsilon} k\delta - e^{-\varepsilon} \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right], \end{aligned}$$

which further leads to

$$\begin{aligned} &\mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right] - \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} [\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)})] \right] \\ &\leq e^{-\varepsilon} k\delta - e^{-\varepsilon} \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right] + \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right] \\ &\leq 1 - e^{-\varepsilon} + e^{-\varepsilon} k\delta. \end{aligned}$$

The other side of the inequality can be similarly obtained.

The proof is completed. \square

We then prove Theorem 3 based on Theorem 2.

Proof of Theorem 3. By Theorem 2, we have that

$$\mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} [\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)})] \right] \leq e^{-\varepsilon} k\delta + 1 - e^{-\varepsilon} + \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right].$$

Since $\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \geq 0$, we have that for any $\alpha > 0$,

$$\begin{aligned} &\mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} [\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)})] \right] \\ &\geq \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)}) \mathbb{I}_{\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \geq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha} \right] \right] \\ &\geq \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[(\alpha + \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \mathbb{I}_{\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \geq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha} \right] \right]. \end{aligned}$$

Furthermore, by splitting $\mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right]$ into two parts, we have

$$\begin{aligned} &\mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} [\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)})] \right] - \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \right] \right] \\ &\geq \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[(\alpha + \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \mathbb{I}_{\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \geq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha} \right] \right] \\ &\quad - \left(\mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \mathbb{I}_{\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \geq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha} \right] \right] \right. \\ &\quad \left. + \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \mathbb{I}_{\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) < \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha} \right] \right] \right) \\ &= \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\alpha \mathbb{I}_{\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \geq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha} \right] \right] - \mathbb{E}_{S \sim \mathcal{D}^{kN}} \left[\mathbb{E}_{\mathcal{B}(S)} \left[\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) \mathbb{I}_{\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) < \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha} \right] \right] \\ &\geq \alpha \mathbb{P} \left(\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)}) \geq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha \right) - \mathbb{P} \left(\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)}) < \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha \right) \\ &= \alpha \left(1 - \mathbb{P} \left(\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)}) < \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha \right) \right) - \mathbb{P} \left(\mathcal{R}_{\mathcal{D}} (h_{\mathcal{B}(S)}) < \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}} (h_{\mathcal{B}(S)}) + \alpha \right). \end{aligned}$$

By simple rearranging, we have

$$\mathbb{P}\left(\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \leq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}}(h_{\mathcal{B}(S)}) + \alpha\right) \geq \frac{\alpha - (e^{-\varepsilon}k\delta + 1 - e^{-\varepsilon})}{1 + \alpha}. \quad (12)$$

We then calculate the right side of eq.(12) according to the two cases mentioned in Lemma 3:

(1). If $\varepsilon \leq \frac{17}{50}$, let $\alpha = e^{-\varepsilon}k\delta + 1.7\varepsilon$. We then have

$$\frac{\alpha - (e^{-\varepsilon}k\delta + 1 - e^{-\varepsilon})}{1 + \alpha} \geq \frac{\alpha - (e^{-\varepsilon}k\delta + \varepsilon)}{1 + \alpha} = \frac{0.7\varepsilon}{1 + e^{-\varepsilon}k\delta + 1.7\varepsilon} \geq \frac{0.7\varepsilon}{1 + (1.7 + \frac{1}{1.7})\varepsilon} \geq \frac{50}{127}\varepsilon.$$

(2). If $\varepsilon > \frac{17}{50}$, let $\alpha = e^{-\varepsilon}k\delta + 1.1(1 - e^{-\varepsilon})$. We then have

$$\frac{\alpha - (e^{-\varepsilon}k\delta + 1 - e^{-\varepsilon})}{1 + \alpha} = \frac{0.1(1 - e^{-\varepsilon})}{1 + e^{\varepsilon}k\delta + 1.1(1 - e^{-\varepsilon})} \geq \frac{0.1(1 - e^{-\varepsilon})}{1 + 1.19(1 - e^{-\varepsilon})} \geq \frac{1 - e^{-\varepsilon}}{219}.$$

The proof is completed. \square

Before moving on to the proof of Lemma 1, we present a basic lemma about the distance between the output of an exponential mechanism (defined as Definition 4) and the maximum value of the corresponding utility function:

Lemma 4 (c.f. Corollary 3.12, [Dwork and Roth, 2014]). *For any fixed sample set S and an exponential mechanism $\mathcal{E}(S, u, \mathcal{I}, \varepsilon)$, we have*

$$\mathbb{P}\left[u(S, \mathcal{E}(S, u, \mathcal{I}, \varepsilon)) \leq \max_{i \in \mathcal{I}} u(S, i) - \frac{2\Delta u}{\varepsilon}(\ln(|\mathcal{I}|) + t)\right] \leq e^{-t}.$$

We are now ready to prove Lemma 1.

Proof of Lemma 1. We prove the lemma respectively for the two cases: (1). $\varepsilon \leq \frac{1}{5}$, and (2). $\varepsilon > \frac{1}{5}$.

Case 1: Construct algorithm \mathcal{B} with input $S = \{S_i\}_{i=1}^k$ (where $S_i \in \mathcal{Z}^N$) as follows:

Step 1. Run \mathcal{A} on S_i , $i = 1, \dots, k$. Denote the output as $h_i = \mathcal{A}(S_i)$.

Step 2. Let utility function as $u_{h_{[k]}}(S, i) = N \left(\hat{\mathcal{R}}_{S_i}(h_i) - \mathcal{R}_{\mathcal{D}}(h_i) \right)$. Apply the utility $u_{h_{[k]}}$ to an 0.7ε -differential private exponential mechanism $\mathcal{E}(S, u_{h_{[k]}}, [k], \varepsilon)$ and return the output.

By the classical composition property of differential privacy, one can easily obtain that \mathcal{B} is $(1.7\varepsilon, \delta)$ differentially private.

We then prove that \mathcal{B} satisfies

$$\mathbb{P}\left[\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \leq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}}(h_{\mathcal{B}(S)}) + ke^{-2\varepsilon}\delta + 6\varepsilon\right] < \varepsilon.$$

Since eq.(4) holds, we have that

$$\mathbb{P}\left(\forall i, \mathcal{R}_{\mathcal{D}}(\mathcal{A}(S_i)) \leq 4\varepsilon + \hat{\mathcal{R}}_{S_i}(\mathcal{A}(S_i))\right) \leq \left(1 - \frac{e^{-1.7\varepsilon}\delta}{\varepsilon} \ln\left(\frac{2}{\varepsilon}\right)\right)^k,$$

which leads to

$$\mathbb{P}\left(\exists i, \mathcal{R}_{\mathcal{D}}(\mathcal{A}(S_i)) > 4\varepsilon + \hat{\mathcal{R}}_{S_i}(\mathcal{A}(S_i))\right) > 1 - \left(1 - \frac{1}{k} \ln\left(\frac{2}{\varepsilon}\right)\right)^k \stackrel{(*)}{\geq} 1 - \frac{\varepsilon}{2}, \quad (13)$$

where eq.(*) is due to $1 + x \leq e^x$.

On the other hand, the sensitivity of $u_{h_{[k]}}$ for any fixed $h_{[k]}$ can then be bounded as

$$\begin{aligned} & \max_i \max_{S \text{ and } S' \text{ adjacent}} \left| N \left(\hat{\mathcal{R}}_{S_i}(h_i) - \mathcal{R}_{\mathcal{D}}(h_i) \right) - N \left(\hat{\mathcal{R}}_{S'_i}(h_i) - \mathcal{R}_{\mathcal{D}}(h_i) \right) \right| \\ &= \max_i \max_{S \text{ and } S' \text{ adjacent}} \left| N \left(\hat{\mathcal{R}}_{S_i}(h_i) - \hat{\mathcal{R}}_{S'_i}(h_i) \right) \right| \\ &= \max_i \max_{S \text{ and } S' \text{ adjacent}} \left| N \left(\hat{\mathcal{R}}_{S_i}(h_i) - \hat{\mathcal{R}}_{S'_i}(h_i) \right) \right| \leq 1. \end{aligned}$$

Therefore, due to Lemma 4, for any fixed $h_{[k]}$ and any $\alpha \in \mathbb{R}$, we have

$$\mathbb{P} \left[u_{h_{[k]}}(S, \mathcal{E}(S, u_{h_{[k]}}, \mathcal{I}, \varepsilon)) \leq \max_{i \in \mathcal{I}} u_{h_{[k]}}(S, i) - \alpha \right] \leq |\mathcal{I}| e^{-\frac{0.7\alpha\varepsilon}{2}}.$$

We further set $\alpha = 0.11N\varepsilon$ and have

$$\mathbb{P} \left[\text{Gen}_{\mathcal{E}(S, u_{h_{[k]}}, \mathcal{I}, \varepsilon)}^{h_{\mathcal{E}(S, u_{h_{[k]}}, \mathcal{I}, \varepsilon)}} \leq \max_{i \in \mathcal{I}} \text{Gen}_{S_i, h_i} - \varepsilon \right] \leq |\mathcal{I}| e^{-\frac{0.077N\varepsilon^2}{2}} \leq \varepsilon \left(\frac{85}{127} - \frac{1}{2} \right). \quad (14)$$

Combing eq.(13) and eq.(14), we have

$$\mathbb{P} \left(\text{Gen}_{S_{i_{\mathcal{B}(S)}}, h_{\mathcal{B}(S)}} > 1.7 \times 1.7\varepsilon + \varepsilon \right) > 1 - \frac{85}{127}\varepsilon,$$

which completes the proof by $\varepsilon = ke^{-1.7\varepsilon}\delta$.

Case 2: We construct \mathcal{B} the same as **Case 1** except setting the privacy parameter as $-\ln(0.9)$. Similar to **Case 1**, we have

$$\mathbb{P} \left(\exists i, \mathcal{R}_{\mathcal{D}}(\mathcal{A}(S_i)) > 1.2(1 - 0.9e^{-\varepsilon}) + \hat{\mathcal{R}}_{S_i}(\mathcal{A}(S_i)) \right) > 1 - \left(1 - \frac{1}{k} \ln \left(\frac{220}{1 - 0.9e^{-\varepsilon}} \right) \right)^k \stackrel{(*)}{\geq} 1 - \frac{1 - 0.9e^{-\varepsilon}}{220}, \quad (15)$$

and

$$\mathbb{P} \left[\text{Gen}_{\mathcal{E}(S, u_{h_{[k]}}, \mathcal{I}, \varepsilon)}^{h_{\mathcal{E}(S, u_{h_{[k]}}, \mathcal{I}, \varepsilon)}} \leq \max_{i \in \mathcal{I}} \text{Gen}_{S_i, h_i} - 0.01(1 - 0.9e^{-\varepsilon}) \right] \leq |\mathcal{I}| e^{0.01 \frac{N \ln(0.9)(1 - 0.9e^{-\varepsilon})}{2}} \leq \frac{1 - 0.9e^{-\varepsilon}}{220 \times 219}. \quad (16)$$

Combining eq.(15) and eq.(16) completes the proof of **Case 2**. □

A.2 PROOFS OF THEOREM 1

We use Theorem 3 and Lemma 1 to derive the proof of Theorem 1.

Proof of Theorem 1. We prove Theorem 1 by reduction to absurdity. We only prove $\mathbb{P}(\text{Gen}_{S, \mathcal{A}(S)} > a) < \frac{b}{2}$, while $\mathbb{P}(\text{Gen}_{S, \mathcal{A}(S)} < -a) < \frac{b}{2}$, can be proven following the same routine by reversing the order of $\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)})$ and $\hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}}(h_{\mathcal{B}(S)})$ in Lemma 1.

Suppose there exists an algorithm \mathcal{A} satisfies (ε, δ) differential privacy such that either the bound for $\varepsilon \leq \frac{1}{5}$ or $\varepsilon > \frac{1}{5}$ in Theorem 1 is false.

Then, by Theorem 3, for a multi-sample-set $(1.7\varepsilon, \delta)$ -DP preserving algorithm, if $1.7\varepsilon \leq \frac{17}{50}$, by letting $k = \frac{1.7\varepsilon}{1.7\varepsilon - 1.7\varepsilon\delta} = \frac{\varepsilon}{\varepsilon - 1.7\varepsilon\delta}$, we have

$$\mathbb{P} \left(\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \leq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}}(h_{\mathcal{B}(S)}) + \varepsilon + 1.7 \times 1.7\varepsilon \right) \geq \frac{85\varepsilon}{127}, \quad (17)$$

which contradicts the conclusion of Lemma 1 for $\varepsilon \leq \frac{1}{5}$. On the other hand, if $\varepsilon > \frac{1}{5}$, we have

$$\mathbb{P} \left(\mathcal{R}_{\mathcal{D}}(h_{\mathcal{B}(S)}) \leq \hat{\mathcal{R}}_{S_{i_{\mathcal{B}(S)}}}(h_{\mathcal{B}(S)}) + 1.2(1 - 0.9e^{-\varepsilon}) \right) \geq \frac{1 - 0.9e^{-\varepsilon}}{219}, \quad (18)$$

which contradicts the conclusion of Lemma 1 for $\varepsilon > \frac{1}{5}$.

The proof is completed. □

B PROOFS OF COMPOSITION THEOREMS

This section proves the composition theorems. It is organized as follows: Section B.1 shows several definitions and lemmas which will be used throughout the proof; Section B.2 proves a preparation lemma on the KL divergence $D_{KL}(\mathcal{A}(S) \parallel \mathcal{A}(S'))$ between the hypotheses $\mathcal{A}(S)$ and $\mathcal{A}(S')$; based on this lemma Section B.3 proves a composition theorem of ε -differential privacy; Section B.4 extends the composition theorem to (ε, δ) -differential privacy; Section B.5 further tightens the estimate of δ' under some assumptions; and Section B.6 analyses the tightness of this estimation.

B.1 PRELIMINARIES

In this section, we define several measures between random variables that is omitted in Lemma 3, and further presents several lemmas that will be used in the proof of composition theorems.

Definition 5 (Max Divergence; cf. [Dwork and Roth, 2014], Definition 3.6). *For any random variables X and Y , the max divergence between X and Y is defined as*

$$D_\infty(X\|Y) = \max_{U \subseteq \text{Supp}(X)} \left[\log \frac{\mathbb{P}(X \in U)}{\mathbb{P}(Y \in U)} \right].$$

Definition 6 (δ -Approximate Max Divergence; cf. [Dwork and Roth, 2014], Definition 3.6). *For any random variables X and Y , the δ -approximate max divergence between X to Y is defined as*

$$D_\infty^\delta(X\|Y) = \max_{U \subseteq \text{Supp}(X): \mathbb{P}(Y \in U) \geq \delta} \left[\log \frac{\mathbb{P}(X \in U) - \delta}{\mathbb{P}(Y \in U)} \right].$$

Definition 7 (Statistical Distance; cf. [Dwork and Roth, 2014]). *For any random variables X and Y , the statistical distance between X and Y is defined as*

$$\Delta(X\|Y) = \max_U |\mathbb{P}(X \in U) - \mathbb{P}(Y \in U)|.$$

We then recall the following lemma which shows for any two distributions, there exist another two distributions with the same max-divergence and order-invariant KL divergence.

Lemma 5 (cf. [Dwork and Rothblum, 2016], Lemmas 3.9 and 3.10). *For any two distributions P and P' , there exist distributions Q and Q' such that*

$$\max\{D_\infty(Q\|Q'), D_\infty(Q'\|Q)\} = \max\{D_\infty(P\|P'), D_\infty(P'\|P)\},$$

and

$$D_{KL}(P\|P') \leq D_{KL}(Q\|Q') = D_{KL}(Q'\|Q).$$

Azuma Lemma [Boucheron et al., 2013] on concentration inequality of martingales will also be used in the proof of composition theorems.

Lemma 6 (Azuma Lemma; cf. [Mohri et al., 2018], p. 371). *Suppose $\{Y_i\}_{i=1}^T$ is a sequence of random variables, where $Y_i \in [-a_i, a_i]$. Let $\{X_i\}_{i=1}^T$ be a sequence of random variables such that,*

$$\mathbb{E}(Y_i | X_{i-1}, \dots, X_1) \leq C_i,$$

where $\{C_i\}_{i=1}^T$ is a sequence of constant real numbers. Then, we have the following inequality,

$$\mathbb{P} \left(\sum_{i=1}^T Y_i \geq \sum_{i=1}^T C_i + t \sqrt{\sum_{i=1}^T a_i^2} \right) \leq e^{-\frac{t^2}{2}}.$$

When proving the composition bound of Theorem 4 and Theorem 6, we will need to calculate the form of the (maximum) optimal point of function $f(\{\alpha_i\}_{i=1}^T) \triangleq 1 - \prod_{i=1}^T (1 - \alpha_i A_i)$ under some constraints. We put the form into the following lemma.

Lemma 7. *The maximum of function $f(\{\alpha_i\}_{i=1}^T) = 1 - \prod_{i=1}^T (1 - \alpha_i A_i)$ over $\Omega\{1 \leq \alpha_i \leq c_i, \text{ and } \prod_{i=1}^T \alpha_i = c\}$ (c_i, A_i is fixed positive real and $c_i A_i \leq 1$) is achieved at $\partial\Omega$.*

The proof can be derived by simple reduction to absurdity, and we omit it here.

B.2 PROOF OF LEMMA 2

Proof of Lemma 2. By Lemma 5, we have a random variable $\mathcal{M}(S)$ and $\mathcal{M}(S')$, which satisfies

$$D_\infty(\mathcal{M}(S)\|\mathcal{M}(S')) \leq \varepsilon, \quad D_\infty(\mathcal{M}(S')\|\mathcal{M}(S)) \leq \varepsilon,$$

and

$$D_{KL}(\mathcal{A}(S)\|\mathcal{A}(S')) \leq D_{KL}(\mathcal{M}(S)\|\mathcal{M}(S')) = D_{KL}(\mathcal{M}(S')\|\mathcal{M}(S)). \quad (19)$$

Therefore, we only need to derive a bound for $D_{KL}(\mathcal{M}(S)\|\mathcal{M}(S'))$.

By direct calculation,

$$\begin{aligned} & D_{KL}(\mathcal{M}(S)\|\mathcal{M}(S')) \\ \stackrel{(*)}{=} & \frac{1}{2} [D_{KL}(\mathcal{M}(S)\|\mathcal{M}(S')) + D_{KL}(\mathcal{M}(S')\|\mathcal{M}(S))] \\ = & \frac{1}{2} \int \log \frac{d\mathbb{P}(\mathcal{M}(S))}{d\mathbb{P}(\mathcal{M}(S'))} d\mathbb{P}(\mathcal{M}(S)) + \frac{1}{2} \int \log \frac{d\mathbb{P}(\mathcal{M}(S'))}{d\mathbb{P}(\mathcal{M}(S))} d\mathbb{P}(\mathcal{M}(S')) \\ = & \frac{1}{2} \int \log \frac{d\mathbb{P}(\mathcal{M}(S))}{d\mathbb{P}(\mathcal{M}(S'))} d[\mathbb{P}(\mathcal{M}(S)) - \mathbb{P}(\mathcal{M}(S'))] \\ & + \frac{1}{2} \int \left(\log \frac{d\mathbb{P}(\mathcal{M}(S'))}{d\mathbb{P}(\mathcal{M}(S))} + \log \frac{d\mathbb{P}(\mathcal{M}(S))}{d\mathbb{P}(\mathcal{M}(S'))} \right) d\mathbb{P}(\mathcal{M}(S')) \\ = & \frac{1}{2} \int \log \frac{d\mathbb{P}(\mathcal{M}(S))}{d\mathbb{P}(\mathcal{M}(S'))} d[\mathbb{P}(\mathcal{M}(S)) - \mathbb{P}(\mathcal{M}(S'))] + \frac{1}{2} \int \log 1 d\mathbb{P}(\mathcal{M}(S')) \\ = & \frac{1}{2} \int \log \frac{d\mathbb{P}(\mathcal{M}(S))}{d\mathbb{P}(\mathcal{M}(S'))} d[\mathbb{P}(\mathcal{M}(S)) - \mathbb{P}(\mathcal{M}(S'))], \end{aligned} \quad (20)$$

where eq. (*) comes from eq. (19).

We now analyse the last integration in eq. (20). Define

$$k(y) \triangleq \frac{d\mathbb{P}(\mathcal{M}(S) = y)}{d\mathbb{P}(\mathcal{M}(S') = y)} - 1. \quad (21)$$

Therefore,

$$k(y)d\mathbb{P}(\mathcal{M}(S') = y) = d\mathbb{P}(\mathcal{M}(S) = y) - d\mathbb{P}(\mathcal{M}(S') = y). \quad (22)$$

Additionally,

$$\begin{aligned} \mathbb{E}_{\mathcal{M}(S')} k(\mathcal{M}(S')) &= \int_{y \in \mathcal{H}} k(y) d\mathbb{P}(\mathcal{M}(S') = y) \\ &= \int_{y \in \mathcal{H}} d(\mathbb{P}(\mathcal{M}(S) = y) - d\mathbb{P}(\mathcal{M}(S') = y)) \\ &= 0. \end{aligned} \quad (23)$$

By calculating the integration of the both sides of eq. (22), we have

$$\int k(y) d\mathbb{P}(\mathcal{M}(S') = y) = 0.$$

Also, combined with the definition of $k(y)$ (see eq. 21), the right-hand side (RHS) of eq. (20) becomes

$$\text{RHS} = \mathbb{E}_{\mathcal{M}(S')} k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1). \quad (24)$$

Since M is ε -differentially private, $k(y)$ is bounded from both sides as follows,

$$e^{-\varepsilon} - 1 \leq k(y) \leq e^\varepsilon - 1. \quad (25)$$

We now calculate the maximum of eq. (24) subject to eqs. (23) and (25).

First, we argue that the maximum is achieved when $k(\mathcal{M}(S')) \in \{e^{-\varepsilon} - 1, e^\varepsilon - 1\}$ with probability 1 (almost surely). When $k(\mathcal{M}(S')) \in \{e^{-\varepsilon} - 1, e^\varepsilon - 1\}$, almost surely, the distribution for $k(\mathcal{M}(S'))$ is as following,

$$\begin{aligned}\mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1) &= \frac{1}{1 + e^\varepsilon}, \\ \mathbb{P}^*(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1) &= \frac{e^\varepsilon}{1 + e^\varepsilon}.\end{aligned}$$

We argue that it is the distribution that maximizes $k(\mathcal{M}(S'))$.

For the brevity, we denote the probability measure for a given distribution Q as \mathbb{P}_Q . Similarly, \mathbb{P}^* corresponds the distribution Q^* . We prove that Q^* maximizes eq. (24) in the following two cases: (1) $\mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) \leq \mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1)$, and (2) $\mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) > \mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1)$

Case 1: $\mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) \leq \mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1)$

We have

$$\begin{aligned}& \mathbb{E}_{\mathcal{M}(S') \sim Q^*}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1)) \\ &= \mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1) \cdot \varepsilon(e^\varepsilon - 1) - \mathbb{P}^*(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1) \cdot \varepsilon(e^{-\varepsilon} - 1) \\ &= (\mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1) - \mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0)) \cdot \varepsilon(e^\varepsilon - 1) \\ &\quad + \mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) \cdot \varepsilon(e^\varepsilon - 1) - \mathbb{P}^*(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1) \cdot \varepsilon(e^{-\varepsilon} - 1) \\ &\geq \mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) \cdot \varepsilon(1 - e^{-\varepsilon}) - \mathbb{P}^*(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1) \cdot \varepsilon(1 - e^{-\varepsilon}) \\ &\quad + \mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) \cdot \varepsilon(e^\varepsilon - 1) - \mathbb{P}^*(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1) \cdot \varepsilon(e^{-\varepsilon} - 1).\end{aligned}$$

Note that

$$\begin{aligned}\mathbb{P}_Q(k(\mathcal{M}(S')) < 0) &= \mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1) - \mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) \\ &\quad + \mathbb{P}^*(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1).\end{aligned}$$

Therefore, together with the condition eq. (25),

$$\begin{aligned}& \mathbb{E}_{\mathcal{M}(S') \sim Q}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1) I_{k(\mathcal{M}(S')) \leq 0}) \\ &\leq (\mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1) - \mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0)) \cdot \varepsilon(1 - e^{-\varepsilon}) \\ &\quad + \mathbb{P}^*(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1) \cdot \varepsilon(1 - e^{-\varepsilon}).\end{aligned}\tag{26}$$

Also,

$$\mathbb{E}_{\mathcal{M}(S') \sim Q}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1) I_{k(\mathcal{M}(S')) > 0}) \leq \mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) \cdot \varepsilon(e^\varepsilon - 1).\tag{27}$$

Therefore, combined inequalities eqs. (26) and (27), we have

$$\mathbb{E}_{\mathcal{M}(S') \sim Q}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1)) \leq \mathbb{E}_{\mathcal{M}(S') \sim Q^*}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1)).$$

Since the distribution Q is arbitrary, the distribution Q^* maximizes the $k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1)$.

Case 2: $\mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) > \mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1)$

We first prove that if $\mathbb{P}_Q(1 - e^{-\varepsilon} < k(\mathcal{M}(S')) < 0) \neq 0$, there exists a distribution Q' such that

$$\begin{aligned}\mathbb{P}_{Q'}(k(\mathcal{M}(S')) \geq 0) &= \mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0), \\ \mathbb{P}_{Q'}(k(\mathcal{M}(S')) < 0) &= \mathbb{P}_Q(k(\mathcal{M}(S')) < 0), \\ \mathbb{P}_{Q'}(k(\mathcal{M}(S')) < 0) &= \mathbb{P}_{Q'}(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1), \\ \mathbb{E}_{Q'}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1)) &> \mathbb{E}_{Q'}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1)),\end{aligned}$$

while the two conditions (eqs. 23, 25) still hold.

Additionally, we have assumed that

$$\mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) > \mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1).$$

Therefore,

$$\mathbb{P}_Q(k(\mathcal{M}(S')) \leq 0) < \mathbb{P}^*(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1).$$

Also, since the distribution Q' is arbitrary, let it satisfy

$$\mathbb{P}_{Q'}(k(\mathcal{M}(S')) < 0) = \mathbb{P}_Q(k(\mathcal{M}(S')) < 0) = \mathbb{P}_{Q'}(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1).$$

Then, in order to meet the condition eq. (23), let

$$\mathbb{P}_{Q'}(k(\mathcal{M}(S')) = e^\varepsilon - 1) > \mathbb{P}_Q(k(\mathcal{M}(S')) = e^\varepsilon - 1),$$

and

$$\mathbb{P}_{Q'}(0 < k(\mathcal{M}(S')) < e^\varepsilon - 1) \leq \mathbb{P}_Q(0 < k(\mathcal{M}(S')) < e^\varepsilon - 1),$$

Since $x \log(x + 1)$ increases when $x > 0$ and decreases when $x < 0$, we have

$$\mathbb{E}_{Q'}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1)) > \mathbb{E}_Q(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1)).$$

Therefore, we have proved that the argument when $\mathbb{P}_Q(k(\mathcal{M}(S')) < 0) \neq \mathbb{P}_Q(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1)$. We now prove the case that

$$\mathbb{P}_Q(k(\mathcal{M}(S')) < 0) = \mathbb{P}_Q(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1),$$

where

$$\mathbb{E}_Q(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1) I_{k(\mathcal{M}(S')) < 0}) = \varepsilon(1 - e^{-\varepsilon}) \mathbb{P}_Q(k(\mathcal{M}(S')) < 0).$$

Applying Jensen's inequality to bound the $\mathbb{E}_Q(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1) I_{k(\mathcal{M}(S')) \geq 0})$, we have

$$\begin{aligned} & \mathbb{E}_Q(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1) I_{k(\mathcal{M}(S')) \geq 0}) \\ &= \mathbb{P}_Q(\mathcal{M}(S') \geq 0) \mathbb{E}_{Q'}(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1) | k(\mathcal{M}(S')) \geq 0) \\ & \stackrel{(*)}{\leq} \mathbb{P}_Q(\mathcal{M}(S') \geq 0) \mathbb{E}_Q(k(\mathcal{M}(S')) | k(\mathcal{M}(S')) \geq 0) \cdot \log(\mathbb{E}_Q(k(\mathcal{M}(S')) | k(\mathcal{M}(S')) \geq 0) + 1), \end{aligned} \quad (28)$$

where the inequality (*) uses Jensen's inequality ($x \log(1 + x)$ is convex with respect to x when $x > 0$). The upper bound in eq. (28) is achieved as long as

$$\mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) = \mathbb{P}_Q(k(\mathcal{M}(S')) = \mathbb{E}_Q(k(\mathcal{M}(S')) | k(\mathcal{M}(S')) \geq 0)).$$

Furthermore,

$$\mathbb{P}_Q(k(\mathcal{M}(S')) < 0) = \mathbb{P}_Q(k(\mathcal{M}(S')) = e^{-\varepsilon} - 1).$$

Therefore, the distribution Q is determined by the cumulative density functions $\mathbb{P}_Q(k(\mathcal{M}(S')) < 0)$ and $\mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0)$.

Hence, maximizing $\mathbb{E}_Q(k(\mathcal{M}(S')) \log(k(\mathcal{M}(S')) + 1))$ is equivalent to maximizing the following object function,

$$g(q) = q(1 - e^{-\varepsilon}) \log e^\varepsilon + (1 - q) \frac{q}{1 - q} (1 - e^{-\varepsilon}) \log \left(\frac{q}{1 - q} (1 - e^{-\varepsilon}) + 1 \right),$$

subject to

$$\frac{q}{1 - q} \leq e^\varepsilon, \quad (29)$$

where $g(q)$ is the maximum of eq. (24) subject to $\mathbb{P}_Q(k(\mathcal{M}(S')) < 0) = q$, and the condition eq. (29) comes from the $\mathbb{P}_Q(k(\mathcal{M}(S')) \geq 0) > \mathbb{P}^*(k(\mathcal{M}(S')) = e^\varepsilon - 1)$ (the assumption of Case 2).

Additionally, $g(q)$ can be represented as follows,

$$q(1 - e^{-\varepsilon}) \log \left(\frac{q}{1-q} (e^\varepsilon - 1) + \varepsilon \right).$$

Since both q and $\frac{q}{1-q}$ monotonously increase, $g(q)$ monotonously increases. Therefore, Q^* maximize eq. (24), which finishes the proof. \square

B.3 PROOF OF THEOREM 5

Based on Lemma 2, we can prove the following composition theorem for ε -differential privacy as a preparation theorem of the general case.

Proof of Theorem 5. We begin by calculating $\log \frac{\mathbb{P}^S(\{W_i\}_{i=0}^T)}{\mathbb{P}^{S'}(\{W_i\}_{i=0}^T)}$ as follows,

$$\begin{aligned} \log \frac{\mathbb{P}^S(\{W_i\}_{i=0}^T)}{\mathbb{P}^{S'}(\{W_i\}_{i=0}^T)} &= \log \left(\prod_{i=0}^T \frac{\mathbb{P}^S(W_i | W_{i-1}, \dots, W_0)}{\mathbb{P}^{S'}(W_i | W_{i-1}, \dots, W_0)} \right) \\ &= \sum_{i=0}^T \log \left(\frac{\mathbb{P}^S(W_i | W_{i-1}, \dots, W_0)}{\mathbb{P}^{S'}(W_i | W_{i-1}, \dots, W_0)} \right) \\ &\stackrel{(*)}{=} \sum_{i=1}^T \log \left(\frac{\mathbb{P}^S(W_i | W_{i-1}, \dots, W_0)}{\mathbb{P}^{S'}(W_i | W_{i-1}, \dots, W_0)} \right) \\ &= \sum_{i=1}^T \log \left(\frac{\mathbb{P}^S(\mathcal{M}_i(W_{i-1}, S) = W_i | W_{i-1}, \dots, W_0)}{\mathbb{P}^{S'}(\mathcal{M}_i(W_{i-1}, S') = W_i | W_{i-1}, \dots, W_0)} \right) \\ &\stackrel{(**)}{=} \sum_{i=1}^T \log \left(\frac{\mathbb{P}^{S, W_{i-1}}(\mathcal{M}_i(W_{i-1}, S) = W_i)}{\mathbb{P}^{S', W_{i-1}}(\mathcal{M}_i(W_{i-1}, S') = W_i)} \right), \end{aligned}$$

where eq. (*) comes from the independence of W_0 with respect to S and eq. (**) is because the independence of \mathcal{M}_i to W_k ($k < i$) when the W_{i-1} is fixed.

By the definition of ε -differential privacy, one has for arbitrary W_{i-1} ,

$$\begin{aligned} D_\infty(\mathcal{M}_i(W_{i-1}, S) \| \mathcal{M}_i(W_{i-1}, S')) &< \varepsilon_i, \\ D_\infty(\mathcal{M}_i(W_{i-1}, S') \| \mathcal{M}_i(W_{i-1}, S)) &< \varepsilon_i. \end{aligned}$$

Thus, by Lemma 2, we have that

$$\begin{aligned} &\mathbb{E}^S \left(\log \left(\frac{\mathbb{P}(\mathcal{M}_i(W_{i-1}, S) = W_i)}{\mathbb{P}(\mathcal{M}_i(W_{i-1}, S') = W_i)} \right) \middle| W_{i-1}, \dots, W_0 \right) \\ &= D_{KL}(\mathcal{M}_i(W_{i-1}, S) \| \mathcal{M}_i(W_{i-1}, S')) \\ &\leq \varepsilon_i \frac{e^{\varepsilon_i} - 1}{e^{\varepsilon_i} + 1}. \end{aligned} \tag{30}$$

Combining Azuma Lemma (Lemma 6) and eq. (30), we can finally derive the following equation

$$\mathbb{P}^S \left(\left\{ \frac{\mathbb{P}^S(W_i = W'_i, i \in \{0, \dots, T\})}{\mathbb{P}(W_i = W'_i, i \in \{0, \dots, T\})} > e^{\varepsilon'} \right\} \right) < \delta',$$

where S and S' are adjacent sample sets.

Therefore, the algorithm \mathcal{A} is ε' -differentially private.

The proof is completed. \square

B.4 PROOF OF THEOREM 6

Now, we can prove our composition theorems for (ε, δ) -differential privacy. We first prove a composition algorithm of (ε, δ) -differential privacy whose estimate of ε' is somewhat looser than the existing results. Then, we tighten the results and obtain a composition theorem that strictly tighter than the current estimate.

Proof of Theorem 6. It has been proved that the optimal privacy preservation can be achieved by a sequence of independent iterations (see [Kairouz et al., 2017], Theorem 3.5). Therefore, without loss of generality, we assume that the iterations in our theorem are independent with each other.

Fixed any two adjacent sample sets S and S' , and rewrite $W_i(S)$ as W_i^0 , and $W_i(S')$ as W_i^1 . Then, by Lemma 3, for $i \geq 1$ there exist random variables \tilde{W}_i^0 and \tilde{W}_i^1 , such that

$$\Delta \left(W_i^0 \| \tilde{W}_i^0 \right) \leq \frac{\delta_i}{1 + e^{\varepsilon_i}}, \quad (31)$$

$$\Delta \left(W_i^1 \| \tilde{W}_i^1 \right) \leq \frac{\delta_i}{1 + e^{\varepsilon_i}}, \quad (32)$$

$$D_\infty \left(\tilde{W}_i^0 \| \tilde{W}_i^1 \right) \leq \varepsilon_i, \quad (33)$$

$$D_\infty \left(\tilde{W}_i^1 \| \tilde{W}_i^0 \right) \leq \varepsilon_i. \quad (34)$$

Applying Theorem 6 (here, $\delta = \tilde{\delta}$), we have that

$$D_\infty^{\tilde{\delta}} \left(\{\tilde{W}_i^0\}_{i=0}^T \| \{\tilde{W}_i^1\}_{i=0}^T \right) \leq \varepsilon',$$

$$D_\infty^{\tilde{\delta}} \left(\{\tilde{W}_i^1\}_{i=0}^T \| \{\tilde{W}_i^0\}_{i=0}^T \right) \leq \varepsilon'.$$

Apparently, for any sequence of hypothesis sets $\mathcal{H}_0, \dots, \mathcal{H}_T$,

$$\mathbb{P}(W_i^0 \in \mathcal{H}_i) - \min \left\{ \frac{\delta_i}{1 + e^{\varepsilon_i}}, \mathbb{P}(W_i^0 \in \mathcal{H}_i) \right\} \geq 0.$$

Therefore,

$$\begin{aligned} & \mathbb{P}(W_0^0 \in \mathcal{H}_0) \left(\mathbb{P}(W_1^0 \in \mathcal{H}_1) - \min \left\{ \frac{\delta_1}{1 + e^{\varepsilon_1}}, \mathbb{P}(W_1^0 \in \mathcal{H}_1) \right\} \right) \\ & \dots \left(\mathbb{P}(W_T^0 \in \mathcal{H}_T) - \min \left\{ \frac{\delta_T}{1 + e^{\varepsilon_T}}, \mathbb{P}(W_T^0 \in \mathcal{H}_T) \right\} \right) \\ & \leq \mathbb{P}(\tilde{W}_0^0 \in \mathcal{H}_0) \dots \mathbb{P}(\tilde{W}_T^0 \in \mathcal{H}_T) \\ & \leq e^{\varepsilon'} \mathbb{P}(\tilde{W}_0^1 \in \mathcal{H}_0) \dots \mathbb{P}(\tilde{W}_T^1 \in \mathcal{H}_T) + \tilde{\delta}. \end{aligned} \quad (35)$$

Furthermore, by eq. (34), we also have that

$$\mathbb{P}(\tilde{W}_i^0 \in \mathcal{H}_i) \leq \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i).$$

Therefore,

$$\mathbb{P}(\tilde{W}_0^0 \in \mathcal{H}_0) \dots \mathbb{P}(\tilde{W}_n^0 \in \mathcal{H}_T) \leq \prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(\tilde{W}_0^1 \in \mathcal{H}_0) \dots \mathbb{P}(\tilde{W}_T^1 \in \mathcal{H}_T) + \tilde{\delta}.$$

Then, we prove this theorem in two cases: (1) $\prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \leq e^{\varepsilon'}$; and (2)

$$\prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} > e^{\varepsilon'}.$$

Case 1- $\prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \leq e^{\varepsilon'}$.

We have that

$$\begin{aligned} & \mathbb{P}(\tilde{W}_0^1 \in \mathcal{H}_0) \left(\mathbb{P}(\tilde{W}_1^1 \in \mathcal{H}_1) - \frac{\delta_1}{1 + e^{\varepsilon_1}} \right) \cdots \left(\mathbb{P}(\tilde{W}_T^1 \in \mathcal{H}_T) - \frac{\delta_T}{1 + e^{\varepsilon_T}} \right) \\ & \leq \mathbb{P}(W_0^1 \in \mathcal{H}_0) \cdots \mathbb{P}(W_T^1 \in \mathcal{H}_T). \end{aligned}$$

By simple calculation, we have that

$$\begin{aligned} & \prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(\tilde{W}_0^1 \in \mathcal{H}_0) \cdots \mathbb{P}(\tilde{W}_T^1 \in \mathcal{H}_T) \\ & \leq \prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(W_0^1 \in \mathcal{H}_0) \cdots \mathbb{P}(W_T^1 \in \mathcal{H}_T) \\ & \quad + \prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(\tilde{W}_0^1 \in \mathcal{H}_0) \cdots \mathbb{P}(\tilde{W}_T^1 \in \mathcal{H}_T) \\ & \quad - \prod_{i=1}^n \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(\tilde{W}_0^1 \in \mathcal{H}_0) \\ & \quad \left(\mathbb{P}(\tilde{W}_1^1 \in \mathcal{H}_1) - \frac{\delta_1}{1 + e^{\varepsilon_1}} \right) \cdots \left(\mathbb{P}(\tilde{W}_T^1 \in \mathcal{H}_T) - \frac{\delta_T}{1 + e^{\varepsilon_T}} \right). \end{aligned}$$

Apparently,

$$\min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(\tilde{W}_0^1 \in \mathcal{H}_i) \leq 1,$$

and when $A > B$, $f(x) = Ax - (x - a)B$ increases when x increases.

Therefore, we have that

$$\begin{aligned} & \prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(\tilde{W}_0^1 \in \mathcal{H}_0) \cdots \mathbb{P}(\tilde{W}_T^1 \in \mathcal{H}_T) \\ & - \prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \mathbb{P}(W_0^1 \in \mathcal{H}_0) \\ & \quad \left(\mathbb{P}(\tilde{W}_1^1 \in \mathcal{H}_1) - \frac{\delta_1}{1 + e^{\varepsilon_1}} \right) \cdots \left(\mathbb{P}(\tilde{W}_T^1 \in \mathcal{H}_T) - \frac{\delta_T}{1 + e^{\varepsilon_T}} \right) \\ & \leq 1 - \prod_{i=1}^T \left(1 - \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \frac{\delta_i}{1 + e^{\varepsilon_i}} \right). \end{aligned}$$

Combining with eq. (35), we have that

$$\delta' \leq 1 - \prod_{i=1}^T \left(1 - \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} \frac{\delta_i}{1 + e^{\varepsilon_i}} \right) + 1 - \prod_{i=1}^T \left(1 - \frac{\delta_i}{1 + e^{\varepsilon_i}} \right) + \tilde{\delta}.$$

Case 2- $\prod_{i=1}^T \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\} > e^{\varepsilon'}$:

There exists a sequence of reals $\{\alpha_i\}_{i=1}^T$ such that

$$\begin{aligned} e^{\alpha_i} & \leq \min \left\{ e^{\varepsilon_i}, \frac{1}{\mathbb{P}(\tilde{W}_i^1 \in \mathcal{H}_i)} \right\}, \\ \sum_{i=1}^T \alpha_i & = \varepsilon'. \end{aligned}$$

Therefore, similar to Case 1, we have that

$$\delta' \leq 1 - \prod_{i=1}^T \left(1 - e^{\alpha_i} \frac{\delta_i}{1 + e^{\varepsilon_i}}\right) + 1 - \prod_{i=1}^T \left(1 - \frac{\delta_i}{1 + e^{\varepsilon_i}}\right).$$

Overall, we have proven that

$$\delta' \leq 1 - \prod_{i=1}^T \left(1 - e^{\alpha_i} \frac{\delta_i}{1 + e^{\varepsilon_i}}\right) + 1 - \prod_{i=1}^T \left(1 - \frac{\delta_i}{1 + e^{\varepsilon_i}}\right),$$

where $\sum_{i=1}^T \alpha_i \leq \varepsilon'$ and $\alpha_i \leq \varepsilon_i$.

From Lemma 7, the minimum is realised on the boundary, which is exactly this theorem claims.

The proof is completed. □

Then, we can prove Theorem 4.

Proof of Theorem 4. Applying Theorem 3.5 in [Kairouz et al., 2017] and replacing ε' in the proof of Theorem 6 as

$$\varepsilon' = \min \{\varepsilon'_1, \varepsilon'_2, \varepsilon'_3\}.$$

The proof is completed. □

B.5 PROOF OF COROLLARY 2

Proof of Corollary 2. Let \mathcal{P}_0 and \mathcal{P}_1 be two distributions whose cumulative distribution functions P_0 and P_1 are respectively defined as following:

$$P_0(x) = \begin{cases} \delta, & x = 0 \\ \frac{(1 - \delta)e^\varepsilon}{1 + e^\varepsilon}, & x = 1 \\ \frac{1 - \delta}{1 + e^\varepsilon}, & x = 2 \\ 0, & x = 3 \end{cases},$$

and

$$P_1(x) = \begin{cases} 0, & x = 0 \\ \frac{(1 - \delta)e^\varepsilon}{1 + e^\varepsilon}, & x = 1 \\ \frac{1 - \delta}{1 + e^\varepsilon}, & x = 2 \\ \delta, & x = 3 \end{cases}.$$

By Theorem 3.4 of [Kairouz et al., 2017], the largest magnitude of the (ε', δ') -differential privacy can be calculated from the $\mathcal{P}_0^{\otimes T}$ and $\mathcal{P}_1^{\otimes T}$.

Construct $\tilde{\mathcal{P}}_0$ and $\tilde{\mathcal{P}}_1$, whose cumulative distribution functions are as follows,

$$\tilde{P}_0(x) = \begin{cases} \frac{e^\varepsilon \delta}{1 + e^\varepsilon}, & x = 0 \\ \frac{(1 - \delta)e^\varepsilon}{1 + e^\varepsilon}, & x = 1 \\ \frac{1 - \delta}{1 + e^\varepsilon}, & x = 2 \\ \frac{\delta}{1 + e^\varepsilon}, & x = 3 \end{cases},$$

and

$$\tilde{P}_1(x) \begin{cases} \frac{\delta}{1+e^\varepsilon}, & x=0 \\ \frac{(1-\delta)e^\varepsilon}{1+e^\varepsilon}, & x=1 \\ \frac{1-\delta}{1+e^\varepsilon}, & x=2 \\ \frac{e^\varepsilon\delta}{1+e^\varepsilon}, & x=3 \end{cases}.$$

One can easily verify that

$$\begin{aligned} \Delta(\mathcal{P}_0\|\tilde{\mathcal{P}}_0) &\leq \frac{\delta}{1+e^\varepsilon}, \\ \Delta(\mathcal{P}_1\|\tilde{\mathcal{P}}_1) &\leq \frac{\delta}{1+e^\varepsilon}, \\ D_\infty(\tilde{\mathcal{P}}_0\|\tilde{\mathcal{P}}_1) &\leq \varepsilon, \\ D_\infty(\tilde{\mathcal{P}}_1\|\tilde{\mathcal{P}}_0) &\leq \varepsilon_i. \end{aligned}$$

Let $V_i(x_i) = \log\left(\frac{\tilde{P}_0(x_i)}{\tilde{P}_1(x_i)}\right)$ and $S(x_1, \dots, x_T) = \sum_{i=1}^T V_i(x_i)$.

We have that for any $t > 0$,

$$\mathbb{P}_{\tilde{\mathcal{P}}_0^T}(\{x_i\} : S(\{x_i\}) > \varepsilon') \leq e^{-\varepsilon' t} \mathbb{E}_{\tilde{\mathcal{P}}_0^{\otimes T}}(e^{tS}) = e^{-\varepsilon' t} \left(\frac{e^{t\varepsilon+\varepsilon}}{1+e^\varepsilon} + \frac{e^{-t\varepsilon}}{1+e^\varepsilon} \right)^T = e^{-\varepsilon' t - Tt\varepsilon} \left(\frac{e^{2t\varepsilon+\varepsilon}}{1+e^\varepsilon} + \frac{1}{1+e^\varepsilon} \right)^T. \quad (36)$$

By calculating the derivative,, we have that the minimum of the RHS of eq. (36) is achieved at

$$e^{2\varepsilon t} = e^{-\varepsilon} \frac{T\varepsilon + \varepsilon'}{T\varepsilon - \varepsilon'}. \quad (37)$$

Since $\varepsilon' \geq T \frac{e^\varepsilon - 1}{e^\varepsilon + 1}$,

$$e^{-\varepsilon} \frac{T\varepsilon + \varepsilon'}{T\varepsilon - \varepsilon'} > 1.$$

Therefore, by applying eq.(37) into the RHS of eq. (36), we have that

$$\mathbb{P}_{\tilde{\mathcal{P}}_0^T}(\{x_i\} : S(\{x_i\}) > \varepsilon') \leq e^{-\frac{\varepsilon' + T\varepsilon}{2}} \left(\frac{1}{1+e^\varepsilon} \left(\frac{2T\varepsilon}{T\varepsilon - \varepsilon'} \right) \right)^T \left(\frac{T\varepsilon + \varepsilon'}{T\varepsilon - \varepsilon'} \right)^{-\frac{\varepsilon' + T\varepsilon}{2\varepsilon}}. \quad (38)$$

Define RHS of eq. (38) as δ' . We have $(\tilde{\mathcal{P}}^b)^{\otimes T}$ ($b = 0, 1$) have ε' δ' -approximate max divergence. Then, using similar analysis of the Proof of Theorem 6, we prove this theorem. \square

B.6 TIGHTNESS OF THEOREM 2

This section analyses the tightness of Theorem 2. Specifically, we compare it with our Theorem 4.

In the proof of Theorem 4 (see Section B.3), ε'_3 is derived through Azuma Lemma (Lemma 6). Specifically, the δ' is derived by

$$\begin{aligned} \mathbb{P} \left[S_T \geq \varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1} \right] &\leq e^{-t(\varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1})} \mathbb{E} [e^{tS_T}] \\ &= e^{-t(\varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1})} \mathbb{E}_{\tilde{\mathcal{P}}_0^{\otimes(T-1)}} [e^{tS_{T-1}} \mathbb{E} [e^{tV_T} | x_1, \dots, x_{T-1}]] \\ &\leq e^{-t(\varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1})} \mathbb{E}_{\tilde{\mathcal{P}}_0^{\otimes(T-1)}} [e^{tS_{T-1}}] e^{4t^2\varepsilon^2/8} \\ &\leq e^{-t(\varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1})} e^{Tt^2\varepsilon^2/2}, \end{aligned}$$

where V_i is defined as $\log \frac{\mathbb{P}_{\tilde{\mathcal{P}}_0}(x_i)}{\mathbb{P}_{\tilde{\mathcal{P}}_1}(x_i)} - \mathbb{E}_{\tilde{\mathcal{P}}_0} \left[\log \frac{\mathbb{P}_{\tilde{\mathcal{P}}_0}(x_i)}{\mathbb{P}_{\tilde{\mathcal{P}}_1}(x_i)} \mid x_1, \dots, x_{i-1} \right]$ and S_j is defined as $\sum_{i=1}^j V_i$.

Since $\mathbb{P} \left[S_T \geq \varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1} \right]$ does not depend on t ,

$$\mathbb{P} \left[S_T \geq \varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1} \right] \leq \min_{t>0} e^{-\frac{(\varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1})^2}{2T\varepsilon^2}} = \delta',$$

By contrary, the approach here directly calculates $\mathbb{E}[e^{tS_T}]$, without the shrinkage in the proof of Theorem 4 (see Section B.3). Specifically,

$$e^{-\varepsilon' t - T t \varepsilon} \left(\frac{e^{2t\varepsilon + \varepsilon}}{1 + e^\varepsilon} + \frac{1}{1 + e^\varepsilon} \right)^T = e^{-t\varepsilon} \mathbb{E} [e^{tS_T}] \leq e^{-t(\varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1})} e^{T t^2 \varepsilon^2 / 2}.$$

Therefore,

$$\min_{t>0} e^{-\varepsilon' t - T t \varepsilon} \left(\frac{e^{2t\varepsilon + \varepsilon}}{1 + e^\varepsilon} + \frac{1}{1 + e^\varepsilon} \right)^T \leq \min_{t>0} e^{-t(\varepsilon' - T \frac{e^\varepsilon - 1}{e^\varepsilon + 1})} e^{T t^2 \varepsilon^2 / 2},$$

which leads to

$$e^{-\frac{\varepsilon' + T\varepsilon}{2}} \left(\frac{1}{1 + e^\varepsilon} \left(\frac{2T\varepsilon}{T\varepsilon - \varepsilon'} \right) \right)^T \left(\frac{T\varepsilon + \varepsilon'}{T\varepsilon - \varepsilon'} \right)^{-\frac{\varepsilon' + T\varepsilon}{2\varepsilon}} \leq \delta'.$$

It ensures that this estimate further tightens δ' than Section B.3 (which is also the $\tilde{\delta}$ in Theorem 4) if the ε' is the same.

C SUPPLEMENTARY MATERIALS OF THE APPLICATIONS

This appendix collects the formal description of SGLD and agnostic federated learning, together with the proof for the application in IGMM and the asymptotic generalization bound for agnostic federated learning.

C.1 DETAILED DESCRIPTION OF SGLD AND AGNOSTIC FEDERATED LEARNING

SGLD and agnostic federated learning are described respectively as the following two charts.

Algorithm 2 Stochastic Gradient Langevin Dynamics (SGLD)

Require: Sample $S = \{z_1, \dots, z_N\}$, Gaussian noise variance σ , size of mini-batch τ , iteration steps T , learning rate $\{\eta_1, \dots, \eta_T\}$, loss function $\ell(z, W)$, and the diameter of the gradient space $D \triangleq \max_{W, z, z'} \|\nabla \ell(z, W) - \nabla \ell(z', W)\|$.

- 1: Initialize W_0 randomly.
 - 2: For $t = 1$ to T do:
 - 3: Uniformly sample a mini-batch \mathcal{B}_t of size τ from S without replacement;
 - 4: Sample g_t from $\sigma \mathcal{N}(0, \mathbb{I})$;
 - 5: Update $W_t \leftarrow W_{t-1} - \eta_t \left[\frac{1}{\tau} \sum_{z \in \mathcal{B}_t} \nabla \ell(z, W_{t-1}) + g_t \right]$.
-

Algorithm 3 Differentially Private Federated Learning

Require: Clients $\{c_1, \dots, c_N\}$, Gaussian noise variance σ , size of mini-batch τ , learning rate $\{\eta_1, \dots, \eta_T\}$, iteration steps T , positive constant L .

- 1: Initialize W_0 randomly.
 - 2: For $t = 1$ to T do:
 - 3: Uniformly sample a mini-batch of clients \mathcal{B}_t of size τ without replacement;
 - 4: Randomly sample g_t from $\mathcal{N}(0, L^2 \sigma^2 \mathbb{I})$;
 - 5: Central curator distributes W_{t-1} to the clients in the mini-batch \mathcal{B}_t ;
 - 6: Update $W_t \leftarrow W_{t-1} + \eta_t \left(\frac{1}{\tau} \sum_{c \in \mathcal{B}_t} \frac{\text{ClientUpdate}(c, W_{t-1})}{\max\left(1, \frac{\|\text{ClientUpdate}(c, W_{t-1})\|_2}{L}\right)} + g_t \right)$.
-

By the above two charts, one can easily observe that SGLD is a special case of IGMM with $g = \nabla \ell$, and agnostic federated learning is also a special case of IGMM with $g = \frac{\text{ClientUpdate}}{\max\left(1, \frac{\|\text{ClientUpdate}\|_2}{L}\right)}$ and $D = 2L$.

C.2 PROOF OF THEOREM 7

IGMM applies the sub-sampling technique (i.e., mini-batch) to amplify differential privacy. Therefore, before the proof of Theorem 7, we first present a lemma from [Balle et al., 2018] which provide bound of differential privacy parameters after sub-sampling uniformly without replacement.

Lemma 8 (c.f. Theorem 9, [Balle et al., 2018]). *Let $\mathcal{M}^o : \mathcal{Z}^m \mapsto \Delta \mathcal{H}$ be any mechanism preserving (ε, δ) differential privacy. Let $\mathcal{M}^{wo} : \mathcal{Z}^N \mapsto \Delta \mathcal{Z}^m$ be the uniform sub-sampling without replacement mechanism. Then mechanism $\mathcal{M}^o \circ \mathcal{M}^{wo}$ satisfy $(\log(1 + (m/N)(e^\varepsilon - 1)), m\delta/N)$ differential privacy.*

Proof of Theorem 7. Before the start of the proof, we define several notations. We denote $G_{\mathcal{B}}(W) \triangleq \frac{1}{\|\mathcal{B}\|} \sum_{z \in \mathcal{B}} g(z, W)$ as the mean of g over \mathcal{B} for brevity. We also use \mathbf{p} as the probability density, with \mathbf{p}^V the probability density conditional on any random variable V .

We first calculate the differential privacy of each step. To begin with, step 3 in Algorithm 1 is equivalent to uniformly sampling a mini-batch \mathcal{I}_t from index set $[N]$ with size τ without replacement and letting $\mathcal{B}_t = S_{\mathcal{I}_t}$. Furthermore, for fixed W_{t-1}, \mathcal{I} , and any two adjacent sample sets S and S' , we have

$$\begin{aligned} \frac{\mathbf{p}^{S, \mathcal{I}_t}(W_t = W | W_{t-1})}{\mathbf{p}^{S', \mathcal{I}_t}(W_t = W | W_{t-1})} &= \frac{\mathbf{p}^{S, \mathcal{I}_t}(\eta_t(G_{S_{\mathcal{I}_t}}(W_{t-1}) + \mathcal{N}(0, \sigma^2 \mathbb{I})) = W - W_{t-1})}{\mathbf{p}^{S', \mathcal{I}_t}(\eta_t(G_{S'_{\mathcal{I}_t}}(W_{t-1}) + \mathcal{N}(0, \sigma^2 \mathbb{I})) = W - W_{t-1})} \\ &= \frac{\mathbf{p}^{\mathcal{I}_t, W_{t-1}}(\mathcal{N}(0, \sigma^2 \mathbb{I}) = W')}{\mathbf{p}^{S, S', \mathcal{I}_t, W_{t-1}}(G_{S'_{\mathcal{I}_t}}(W_{t-1}) - G_{S_{\mathcal{I}_t}}(W_{t-1}) + \mathcal{N}(0, \sigma^2 \mathbb{I}) = W')}, \end{aligned}$$

where $\eta_t W' = W - W_{t-1} - \eta_t G_{S_{\mathcal{I}_t}}(W_{t-1})$. Therefore, if $W \sim W_{t-1} + \eta_t(G_{S_{\mathcal{I}_t}}(W_{t-1}) + \mathcal{N}(0, \sigma^2 \mathbb{I}))$, then $W' \sim G_{S_{\mathcal{I}_t}}(W_{t-1}) + \mathcal{N}(0, \sigma^2 \mathbb{I})$. Define

$$D^{S, S', \mathcal{I}_t, W_{t-1}}(W') = \log \frac{\mathbf{p}^{\mathcal{I}_t, W_{t-1}}(\mathcal{N}(0, \sigma^2 \mathbb{I}) = W')}{\mathbf{p}^{S, S', \mathcal{I}_t, W_{t-1}}(G_{S'_{\mathcal{I}_t}}(W_{t-1}) - G_{S_{\mathcal{I}_t}}(W_{t-1}) + \mathcal{N}(0, \sigma^2 \mathbb{I}) = W')},$$

which by the definition of Gaussian distribution further leads to

$$\begin{aligned} D(W') &= -\frac{\|W'\|^2}{2\sigma^2} + \frac{\|W' - G_{S'_{\mathcal{I}_t}}(W_{t-1}) + G_{S_{\mathcal{I}_t}}(W_{t-1})\|^2}{2\sigma^2} \\ &= \frac{2\langle W', -G_{S'_{\mathcal{I}_t}}(W_{t-1}) + G_{S_{\mathcal{I}_t}}(W_{t-1}) \rangle + \|G_{S'_{\mathcal{I}_t}}(W_{t-1}) - G_{S_{\mathcal{I}_t}}(W_{t-1})\|^2}{2\sigma^2}. \end{aligned}$$

Denote $-G_{S'_{\mathcal{I}_t}}(W_{t-1}) + G_{S_{\mathcal{I}_t}}(W_{t-1})$ as \mathbf{v} . By the definition of D , we have that

$$\|\mathbf{v}\| < \frac{1}{\tau} D.$$

On the other hand, since $\langle \mathbf{v}, W' \rangle \sim \mathcal{N}(0, \|\mathbf{v}\|^2 \sigma^2)$, by Chernoff Bound technique,

$$\begin{aligned} \mathbb{P}\left(\langle \mathbf{v}, W' \rangle \geq \frac{\sqrt{2}D\sigma}{\tau} \sqrt{\log \frac{1}{\delta}}\right) &\leq \mathbb{P}\left(\langle \mathbf{v}, W' \rangle \geq \sqrt{2}\|\mathbf{v}\|\sigma \sqrt{\log \frac{1}{\delta}}\right) \\ &\leq \min_t e^{-\sqrt{2}t\|\mathbf{v}\|\sigma \sqrt{\log \frac{1}{\delta}}} \mathbb{E}(e^{t\langle \mathbf{v}, W' \rangle}) \\ &= \delta. \end{aligned}$$

Therefore, with probability at least $1 - \delta$ with respect to W' , we have that

$$D(W') \leq \frac{\sqrt{2}D\sigma \frac{1}{\tau} \sqrt{\log \frac{1}{\delta}} + \frac{1}{\tau^2} D^2}{2\sigma^2}.$$

By Lemma 8, we have that step t is $(\tilde{\varepsilon}, \frac{\tau}{N}\delta)$ -differentially private. Applying Theorem 2 with $\varepsilon = \varepsilon'_3$ we can prove the differential privacy.

By applying Theorem 1, we can prove the generalization bound.

The proof is completed. \square

C.3 PROOF OF COROLLARY 3

Proof of Corollary 3. We first calculate the asymptotic bounds for privacy parameters. Let $\delta = \Theta(1/N^2)$, we have

$$\begin{aligned}
\tilde{\varepsilon} &= \log \left(\frac{N - \tau}{N} + \frac{\tau}{N} \exp \left(\frac{\sqrt{2}D\sigma \frac{1}{\tau} \sqrt{\log \frac{1}{\delta} + \frac{1}{\tau^2} D^2}}{2\sigma^2} \right) \right) \\
&= \log \left(\frac{N - \tau}{N} + \frac{\tau}{N} \exp \left(\Theta(\sqrt{\log N}) \right) \right) \\
&= \log \left(1 + \frac{\tau}{N} \left(\exp \left(\Theta(\sqrt{\log N}) \right) - 1 \right) \right) \\
&= \tilde{\Theta} \left(\frac{1}{N} \right).
\end{aligned} \tag{39}$$

Furthermore, let $\tilde{\delta} = \Theta(\frac{1}{N^2})$. By eq.(39), we have

$$\begin{aligned}
\varepsilon' &= \sqrt{2T \log \left(\frac{1}{\tilde{\delta}} \right)} \tilde{\varepsilon}^2 + T \tilde{\varepsilon} \frac{e^{\tilde{\varepsilon}} - 1}{e^{\tilde{\varepsilon}} + 1} \\
&= \Theta \left(\sqrt{\log N} \tilde{\Theta} \left(\frac{1}{N} \right) \right) + \Theta \left(\tilde{\Theta} \left(\frac{1}{N} \right)^2 \right) = \tilde{\Theta} \left(\frac{1}{N} \right).
\end{aligned}$$

On the other hand, by $\tilde{\delta}' \leq \tilde{\delta} = \mathcal{O}(1/N^2)$, and

$$\begin{aligned}
&2 - \left(1 - \frac{\delta}{1 + e^{\tilde{\varepsilon}}} \right)^T - \left(1 - \frac{\delta e^{\tilde{\varepsilon}}}{1 + e^{\tilde{\varepsilon}}} \right)^{\lceil \frac{N\varepsilon'}{\tau\tilde{\varepsilon}} \rceil} \left(1 - \frac{\delta}{1 + e^{\tilde{\varepsilon}}} \right)^{T - \lceil \frac{N\varepsilon'}{\tau\tilde{\varepsilon}} \rceil} \\
&= \mathcal{O}(\delta) = \mathcal{O} \left(\frac{1}{N^2} \right),
\end{aligned}$$

we have

$$\delta' = 2 - \left(1 - \frac{\delta}{1 + e^{\tilde{\varepsilon}}} \right)^T - \left(1 - \frac{\delta e^{\tilde{\varepsilon}}}{1 + e^{\tilde{\varepsilon}}} \right)^{\lceil \frac{N\varepsilon'}{\tau\tilde{\varepsilon}} \rceil} \left(1 - \frac{\delta}{1 + e^{\tilde{\varepsilon}}} \right)^{T - \lceil \frac{N\varepsilon'}{\tau\tilde{\varepsilon}} \rceil} + \tilde{\delta}' = \mathcal{O} \left(\frac{1}{N^2} \right).$$

When N is large enough, we have \mathcal{A}^{fed} is $(\tilde{\Theta}(1/N), \tilde{\mathcal{O}}(1/N))$ -differentially private, and thus $(\tilde{\Theta}(1/\sqrt{N}), \tilde{\mathcal{O}}(1/N))$ -differentially private. By

$$\frac{2e^{-2\varepsilon'} \delta'}{\varepsilon'} \ln \left(\frac{2}{\varepsilon'} \right) = \frac{2e^{-2\tilde{\Theta}(\frac{1}{\sqrt{N}})} \mathcal{O}(\frac{1}{N^2})}{\tilde{\Theta}(\frac{1}{\sqrt{N}})} \ln \left(\frac{2}{\tilde{\Theta}(\frac{1}{\sqrt{N}})} \right) = \tilde{\mathcal{O}} \left(\frac{1}{N^{\frac{3}{2}}} \right).$$

Since when N is large enough,

$$N \geq \frac{2}{0.077 \tilde{\Theta}(\frac{1}{\sqrt{N}})^2} \ln \left(\frac{43}{254e^{-1.7\tilde{\Theta}(\frac{1}{\sqrt{N}})} \tilde{\mathcal{O}}(\frac{1}{N^{\frac{3}{2}}})} \right),$$

by Theorem 1, the proof is completed. \square